# NOTES ON $E_0$-SEMIGROUPS

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Abstract. The aim and the sole purpose of these notes is to record for future reference that the basic theory of $E_0$-semigroups makes sense in the setting of general semigroups. These notes are based on a set of lectures given by the author at the Indian Statistical Institute, Delhi.

**AMS Classification No. :** Primary 46L55; Secondary 46L99.

Contents



## PREFACE

These notes are based on

(1) a series of a lectures that the author gave at the Indian Statistical Institute, Delhi,

(2) Murugan's thesis,[1]
(3) and Arveson's monograph titled "Non-commutative dynamics and $E$-semigroups".

The theory of $E_0$-semigroups initiated by R.T. Powers and developed extensively by William Arveson in the last 3 decades is very rich and has pleasant connections with probability theory as amply demonstrated by the work of Tsirelson ([35], [36]) and Liebscher ([16]). Roughly speaking, a 1-parameter $E_0$-semigroup is *an action of the additive semigroup* $[0,\infty)$ on a von Neumann algebra. The study of such 1-parameter actions, up to a natural equivalence relation, has received much attention during the last three decades. We refer the reader to [7] for a comprehensive bibliography on the subject and a more thorough treatment. We also recommend the reader to consult [15] for a survey on the 1-parameter theory.

The classification of 1-parameter $E_0$-semigroups on a type I factor is still mysterious. Till now, the only known classification is that of type I $E_0$-semigroups which was due to Arveson and the classification was carried out in his 1989 AMS memoir, "Continuous analogues of Fock space" ([6]). The study of $E_0$-semigroups on type II and type III factors are still at a nascent stage. The work of R. Srinivasan and Oliver Margetts ([17]) is significant in this direction. Recently, there have been efforts to extend the theory of $E_0$-semigroups/$CP$-semigroups to the multivariable context. The notable papers that explore the dilation theoretic aspects of non-commutative dynamics over several variables are [27], [25], [26], [28] and [31].

Mathematically speaking, there is no reason to restrict our attention to the 1-parameter case. The basic theory carries over for general semigroups as well.

(1) Are there good motivations for this generalisation ?
(2) Are there enough examples ? Does understanding these examples result in good mathematics ?
(3) Does one have to use different techniques from other fields to tackle these examples ?

Well, the answer to (1) is, to be really honest, "It is not clear". However the answers to (2) and (3) are a firm "yes". We refer the reader to the paper [4] for the justification of (2) and (3). It is for reasons (2) and (3) that the author believes it is worth investigating beyond the one parameter case.

What is the purpose of these notes ? The one and only purpose of these notes is to record, for future reference, the fact that the basic theory carries over for general semigroups. In particular, we give details of proofs of many results of the 1-parameter

[1] I thank Murugan for being very generous in granting me permission to write Sections 8 and 9 based on his thesis.

theory, which require little modification, in the context of general subsemigroups of locally compact groups (a good example to keep in mind is that of a closed convex cone in a Euclidean space). The reason is when one investigates multiparameter $E_0$-semigroups, it is better to have a reference to point to rather than writing "the details are similar to the one parameter case and left to the reader", especially, if it is repeated several times. These notes serve that purpose.

We also mention some of the results obtained, so far, in the multiparameter context, especially for cones, and refer the reader to the appropriate references wherever we make a mention of it. We would like to mention that this lecture notes is written keeping in mind a typical Indian graduate student who is working in the area of operator algebras. All the Hilbert spaces that we consider are over $\mathbb{C}$ and are separable. Moreover our convention is that the inner product is linear in the first variable.

To repeat, the sole purpose is to put things down in writing and nothing else, as all the ideas are already there in Arveson's monograph. Thus no originality is claimed. If a reference is not given, it should be understood implicitly that the proof is a step by step adaptation of the ideas given in Arveson's monograph. I would like to end this very short introduction by thanking Murugan and Anbu who have helped me immensely in my little understanding of the subject.

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## 1. Normal endomorphisms of $B(\mathcal{H})$

Let $\mathcal{H}$ be a separable Hilbert space[2]. We denote the algebra of bounded operators on $\mathcal{H}$ by $B(\mathcal{H})$ and the set of compact operators by $\mathcal{K}(\mathcal{H})$. The space of trace class operators on $\mathcal{H}$ is denoted by $\mathcal{L}^1(\mathcal{H})$. First let us recall the following facts about trace class operators.

(1) Let $T \in B(\mathcal{H})$ be a positive operator and let $\{\xi_1, \xi_2, \cdots\}$ be an orthonormal basis for $\mathcal{H}$. Then the infinite sum $\sum_{n=1}^{\infty} \langle T\xi_n | \xi_n \rangle$ is called the *trace of* $T$ and is denoted $Tr(T)$. It is a fact that $Tr(T)$ does not depend on the choice of the orthonormal basis.

(2) Let $T \in B(\mathcal{H})$ be given. We say that $T$ is *trace class* if $Tr(|T|)$ is finite where $|T|$ is the square root of $T^*T$. Suppose $T$ is trace class. Then for any orthonormal basis $\{\xi_1, \xi_2, \cdots\}$, the infinite sum $\sum_{n=1}^{\infty} \langle T\xi_n | \xi_n \rangle$ converges and the sum is independent of the chosen basis. We set
$$Tr(T) := \sum_{n=1}^{\infty} \langle T\xi_n | \xi_n \rangle$$
where $\{\xi_1, \xi_2, \cdots\}$ is an orthonormal basis for $\mathcal{H}$. Moreover $Tr(T)$ is called the trace of $T$. We denote the set of trace class operators on $\mathcal{H}$ by $\mathcal{L}^1(\mathcal{H})$.

(3) The set $\mathcal{L}^1(\mathcal{H})$ is a two sided ideal in $B(\mathcal{H})$ and is contained in $\mathcal{K}(\mathcal{H})$. Also the map $\mathcal{L}^1(\mathcal{H}) \ni T \to Tr(T) \in \mathbb{C}$ is linear.

(4) For $T \in \mathcal{L}^1(\mathcal{H})$, set $||T||_1 := Tr(|T|)$. Then $||\ ||_1$ is a norm on $\mathcal{L}^1(\mathcal{H})$ and with respect to this norm $\mathcal{L}^1(\mathcal{H})$ is a separable Banach space.

The following two results are the non-commutative versions of the fact that $c_0^* \cong \ell^1$ and $(\ell^1)^* \cong \ell^\infty$. For $T \in \mathcal{L}^1(\mathcal{H})$, define $\omega_T : \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ by
$$\omega_T(A) = Tr(TA).$$
For $A \in B(\mathcal{H})$, define $\rho_A : \mathcal{L}^1(\mathcal{H}) \to \mathbb{C}$ by
$$\rho_A(T) = Tr(TA).$$

**Theorem 1.1.** *The map*
$$\mathcal{L}^1(\mathcal{H}) \ni T \to \omega_T \in \mathcal{K}(\mathcal{H})^*$$
*is an isometric isomorphism. Also the map*
$$B(\mathcal{H}) \ni A \to \rho_A \in \mathcal{L}^1(\mathcal{H})^*$$
*is an isometric isomorphism.*

[2] All the Hilbert spaces that we consider are assumed to be separable.

For a proof of the above theorem, see [34]. Thus, we can identify $B(\mathcal{H})$ with the dual of $\mathcal{L}^1(\mathcal{H})$ and the weak $*$-topology on $B(\mathcal{H})$, induced via this identification, is called the *$\sigma$-weak topology.*

**Definition 1.2.** *Let $\alpha : B(\mathcal{H}) \to B(\mathcal{H})$ be a $*$-endomorphism. We say that $\alpha$ is normal if $\alpha$ is continuous with respect to the $\sigma$-weak topology.*

We will only consider endomorphisms which preserve the $*$-structure. Thus, we simply call a $*$-endomorphism an endomorphism. A useful fact that enables us to check normality of $*$-endomorphism is given below. We abbreviate convergence in strong operator topology by writing SOT and in weak operator topology by writing WOT.

**Remark 1.3** (Krein-Smulian)**.** *Let $E$ be a separable Banach space and $\phi : E^* \to \mathbb{C}$ be a linear functional. Then $\phi$ is weak $*$-continuous if and only if $\phi$ is weak $*$-sequentially continuous. For a proof, the reader is referred to Corollary 12.8 of* [9].

**Proposition 1.4.** *Let $\alpha : B(\mathcal{H}) \to B(\mathcal{H})$ be a $*$-endomorphism. Then the following statements are equivalent.*

(1) *The endomorphism $\alpha$ is normal.*
(2) *Suppose $A_n \to A$ in WOT. Then $\alpha(A_n) \to \alpha(A)$ in WOT.*

*Proof.* On bounded sets, WOT-convergence and $\sigma$-weak convergence coincide. Thus (1) implies (2). Now suppose (2) holds. Fix $T \in \mathcal{L}^1(\mathcal{H})$. We claim that the map

$$B(\mathcal{H}) \to Tr(\alpha(A)T) \in \mathbb{C}$$

is $\sigma$-weakly continuous.

First assume that $T$ is finite rank. We apply Krein-Smulian theorem. Let $(A_n)$ be a sequence in $B(\mathcal{H})$ such that $A_n \to A$ in the $\sigma$-weak topology. Then $A_n \to A$ in WOT. The hypothesis implies that $\alpha(A_n) \to \alpha(A)$ in WOT. Consequently, the sequence $Tr(\alpha(A_n)T) \to Tr(\alpha(A)T)$. Hence we have proved the claim when $T$ is finite rank. The claim now follows from the fact that finite rank operators are dense in $\mathcal{L}^1(\mathcal{H})$. The proof is now complete. $\square$

Let $d \in \{1, 2, \cdots\} \cup \{\infty\}$ and $\{V_i\}_{i=1}^d$ be a family of isometries with orthogonal range projections, i.e. for $i \neq j$, $V_i^* V_j = 0$. Suppose $d$ is finite. If we define $\alpha : B(\mathcal{H}) \to B(\mathcal{H})$ by the formula:

$$\alpha(A) := \sum_{i=1}^{d} V_i A V_i^*$$

then $\alpha$ is a normal endomorphism. Moreover $\alpha$ is unital if and only if $\sum_{i=1}^d V_i V_i^* = 1$. The content of the next proposition is that $\alpha$ makes sense even if $d$ is infinite.

**Proposition 1.5.** *Let $\{V_i\}_{i=1}^{\infty}$ be a sequence of isometries with orthogonal range projections. For $n \geq 1$, define $\alpha_n : B(\mathcal{H}) \to B(\mathcal{H})$ by the formula*

$$\alpha_n(A) := \sum_{i=1}^{n} V_i A V_i^*.$$

*Then for every $A \in B(\mathcal{H})$, $\alpha_n(A)$ converges in SOT. Let $\alpha : B(\mathcal{H}) \to B(\mathcal{H})$ be defined by*

$$\alpha(A) := \lim_{n \to \infty} \alpha_n(A)$$

*where the limit is taken in the SOT sense. Then $\alpha$ is a normal $*$-endomorphism of $B(\mathcal{H})$. Moreover $\alpha$ is unital if and only if $\sum_{i=1}^{\infty} V_i V_i^* = 1$ in SOT sense.*

*Proof.* Set $P := \lim_{n\to\infty} \sum_{i=1}^{n} V_i V_i^*$ where the limit is taken in the SOT sense. Note that $P$ exists and is a projection. Let $\xi \in \mathcal{H}$ and let $A \in B(\mathcal{H})$ be given. Since $\{V_i\}_{i=1}^{\infty}$ have orthogonal range projections and $V_i$'s are isometries, it follows that

$$||\sum_{i=m+1}^{n} V_i A V_i^* \xi||^2 = \sum_{i=m+1}^{n} ||AV_i^*\xi||^2 \leq ||A||^2 \sum_{i=m+1}^{n} ||V_i^*\xi||^2 \tag{1.1}$$

$$\leq ||A||^2 ||\sum_{i=m+1}^{n} V_i V_i^* \xi||^2. \tag{1.2}$$

Since $\sum_{i=1}^{\infty} V_i V_i^*$ converges strongly, it follows from Eq. 1.1 that for every $A \in B(\mathcal{H})$, $\alpha_n(A)$ converges strongly.

It is clear that $\alpha$ is a $*$-endomorphism. Next we claim that $\alpha$ is normal. Fix $\xi, \eta \in \mathcal{H}$. Define $f_n : B(\mathcal{H}) \to \mathbb{C}$ and $f : B(\mathcal{H}) \to \mathbb{C}$ as follows:

$$f_n(A) : = \langle \alpha_n(A)\xi | \eta \rangle$$
$$f(A) : = \langle \alpha(A)\xi | \eta \rangle.$$

Note that $f_n$ is continuous with respect to the weak operator topology. Estimate 1.1 implies that $f_n$ converges to $f$ uniformly on bounded subsets of $B(\mathcal{H})$. Thus $f$ is continuous on every closed ball equipped with the WOT. The proof is now completed by appealing to Prop. 1.4. $\square$

Our first result, which is fundamental to what follows, is that every normal endomorphism arises in the above fashion. This is essentially a corollary of the representation theory of compact operators which we undertake next.

## 2. The representation theory of $\mathcal{K}(\mathcal{H})$

In this section, we discuss the representation theory of the algebra of compact operators. Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators on $\mathcal{H}$. Let us fix notation. For $\xi, \eta \in \mathcal{H}$, let $\theta_{\xi,\eta}$ be the rank one operator defined by the equation

$$\theta_{\xi,\eta}(\gamma) = \xi\langle\gamma|\eta\rangle.$$

The linear span of $\{\theta_{\xi,\eta} : \xi, \eta \in \mathcal{H}\}$, which is the $*$-algebra of finite rank operators, is dense in $\mathcal{K}(\mathcal{H})$. Observe the following formulas: For $T \in B(\mathcal{H})$, $\xi, \eta, \xi^{'}, \eta^{'} \in \mathcal{H}$,

$$\begin{aligned} T\theta_{\xi,\eta} &= \theta_{T\xi,\eta} \\ \theta_{\xi,\eta}T &= \theta_{\xi,T^*\eta} \\ \theta_{\xi,\eta}\theta_{\xi^{'},\eta^{'}} &= \langle\xi^{'}|\eta\rangle\theta_{\xi,\eta^{'}} \\ \theta_{\xi,\eta}^* &= \theta_{\eta,\xi}. \end{aligned}$$

Choose an orthonormal basis $\{\xi_1, \xi_2, \cdots\}$ and set $E_{ij} := \theta_{\xi_i,\xi_j}$. Then $\{E_{ij}\}_{i,j}$ is a system of "matrix" units, i.e. $E_{ij}E_{kl} = \delta_{jk}E_{il}$ and $E_{ij}^* = E_{ji}$.

**Lemma 2.1.** *The inclusion $\mathcal{K}(\mathcal{H}) \ni T \to T \in B(\mathcal{H})$ defines an irreducible representation of $\mathcal{K}(\mathcal{H})$.*

*Proof.* Let $W \subset \mathcal{H}$ be a non-zero $\mathcal{K}(\mathcal{H})$ invariant subspace. Choose a unit vector $\xi_0 \in W$. Then for $\xi \in \mathcal{H}$, $\xi = \theta_{\xi,\xi_0}(\xi_0) \in W$. This implies that $W = \mathcal{H}$. Hence the proof. $\square$

Let us call the above representation as the identity representation. The crucial facts about the representation theory of compacts are the following:

(1) Any non-degenerate representation of $\mathcal{K}(\mathcal{H})$ is a direct sum of irreducible representations.
(2) The only irreducible representation, up to unitary equivalence, of $\mathcal{K}(\mathcal{H})$ is the identity representation.

This is the content of the next theorem.

**Theorem 2.2.** *Let $\pi : \mathcal{K}(\mathcal{H}) \to B(\widetilde{\mathcal{H}})$ be a non-degenerate representation. Then there exists a Hilbert space $\mathcal{H}_0$ and a unitary $U : \mathcal{H} \otimes \mathcal{H}_0 \to B(\widetilde{\mathcal{H}})$ such that*

$$\pi(A) = U(A \otimes 1)U^*$$

*for $A \in \mathcal{K}(\mathcal{H})$.*

*Proof.* Set $E_n := \sum_{i=1}^{n} E_{ii}$. Note that $E_n$ is an approximate identity of $\mathcal{K}(\mathcal{H})$. Since $\pi$ is non-degenerate, it follows that $\pi(E_n) \to 1$ in SOT. Thus there exists $i$ such that $\pi(E_{ii}) \neq 0$. Choose such an $i$. We claim that $\pi(E_{jj}) \neq 0$ for every $j$. Note that $\pi(E_{ij})$ is a partial isometry with initial space $\pi(E_{jj})$ and final space $\pi(E_{ii}) \neq 0$. Hence $\pi(E_{jj}) \neq 0$. This proves our claim.

Let $\mathcal{H}_0$ be the range space of $\pi(E_{11})$. Denote the dimension of $\mathcal{H}_0$ by $d$ and let $\{\eta_i\}_{i=1}^d$ be an orthonormal basis for $\mathcal{H}_0$. We claim that $\{\pi(E_{i1})\eta_j\}_{i,j}$ is total in $\widetilde{\mathcal{H}}$. Denote the closed linear span of $\{\pi(E_{i1})\eta_j\}_{i,j}$ by $\mathcal{H}_1$. It is clear that $\pi(E_{rs})$ leaves $\mathcal{H}_1$ invariant for every $r, s$. Since the linear span of $\{E_{rs}\}$ is norm dense in $\mathcal{K}(\mathcal{H})$, it follows that $\mathcal{H}_1$ is invariant under $\pi$ and so is $\mathcal{H}_1^\perp$.

Suppose $\mathcal{H}_1^\perp \neq \{0\}$. By definition, it follows that $\mathcal{H}_0 \subset \mathcal{H}_1$. Hence $\mathcal{H}_1^\perp \subset \mathcal{H}_0^\perp = Ker(\pi(E_{11}))$. Thus $\pi(E_{11}) = 0$ on $\mathcal{H}_1^\perp$. But $\pi(E_{i1})$ is a partial isometry with final space $\pi(E_{ii})$ and initial space $\pi(E_{11}) = 0$ on $\mathcal{H}_1^\perp$. Consequently, $\pi(E_{ii}) = 0$ on $\mathcal{H}_1^\perp$ for every $i$ which contradicts the fact that $\pi(E_n) \to 1$ strongly. This proves our claim.

Let $r, s \in \mathbb{N}$ and $j, k \in \{1, 2, \cdots, d\}$ be given. Calculate as follows to observe that

$$\begin{aligned}
\langle \pi(E_{r1})\eta_j | \pi(E_{s1})\eta_k \rangle &= \langle \pi(E_{1s})\pi(E_{r1})\eta_j | \eta_k \rangle \\
&= \delta_{rs} \langle \pi(E_{11})\eta_j | \eta_k \rangle \\
&= \delta_{rs} \langle \eta_j | \eta_k \rangle \\
&= \delta_{rs}\delta_{jk}.
\end{aligned}$$

The above calculation together with the fact that $\{\pi(E_{i1})\eta_j\}_{i,j}$ is total in $\widetilde{\mathcal{H}}$ ensures that there exists a unitary $U : \mathcal{H} \otimes \mathcal{H}_0 \to \widetilde{\mathcal{H}}$ such that $U(\xi_i \otimes \eta_j) = \pi(E_{i1})\eta_j$. A direct calculation reveals that $U(E_{rs} \otimes 1)U^* = \pi(E_{rs})$. Since the linear span of $\{E_{rs} : r, s\}$ is dense in $\mathcal{K}(\mathcal{H})$, it follows that $\pi(A) = U(A \otimes 1)U^*$ for every $A \in \mathcal{K}(\mathcal{H})$. □

**Exercise 2.1.** *Let $\pi : \mathcal{K}(\mathcal{H}) \to B(\widetilde{\mathcal{H}})$ be a non-degenerate representation. Suppose there exists a Hilbert space $\mathcal{H}_0$ and a unitary $U : \mathcal{H} \otimes \mathcal{H}_0 \to \widetilde{\mathcal{H}}$ such that*

$$\pi(A) = U(A \otimes 1)U^*$$

*for $A \in \mathcal{K}(\mathcal{H})$. Show that $dim(\mathcal{H}_0)$ is the dimension of the range space of $\pi(p)$ where $p$ is any rank one projection. The dimension of $\mathcal{H}_0$ is called the **the multiplicity** of the identity representation in $\pi$.*

With the representation theory of the algebra of compact operators in hand, we can now prove that every normal endomorphism of $B(\mathcal{H})$ arises as in Prop. 1.5.

**Theorem 2.3.** *Let $\alpha : B(\mathcal{H}) \to B(\mathcal{H})$ be a non-zero normal $*$-endomorphism. Then there exists $d \in \{1, 2, \cdots\} \cup \{\infty\}$ and isometries $\{V_i\}_{i=1}^d$ with orthogonal range projections such that*

$$\alpha(A) = \sum_{i=1}^{d} V_i A V_i^*$$

*for $A \in B(\mathcal{H})$. When $d$ is infinite, the sum is interpreted in the sense of a strong operator limit.*

*Proof.* Let $E_n := \sum_{i=1}^{n} E_{ii}$. Note that $E_n$ increases to 1 in SOT and hence in $\sigma$-weak topology. Since the endomorphism $\alpha$ is normal, it follows that $\alpha(E_n) \to \alpha(1)$ in the $\sigma$-weak topology. But $\alpha(E_n)$ is an increasing sequence of projections. Thus $\alpha(E_n) \to \alpha(1)$ in SOT. Set $P := \alpha(1)$.

Observe that the range of $\alpha$ is contained in $PB(\mathcal{H})P \cong B(P\mathcal{H})$. Restricting $\alpha$ to $\mathcal{K}(\mathcal{H})$, we obtain a non-degenerate representation of $\mathcal{K}(\mathcal{H})$ on $P\mathcal{H}$. Let $\mathcal{H}_0$ and $U$ be as in Theorem 2.2. Let $d := \dim(\mathcal{H}_0)$ and $\{\eta_i\}_{i=1}^{d}$ be an orthonormal basis for $\mathcal{H}_0$. Define for $i \in \{1, 2, \cdots, d\}$, $V_i : \mathcal{H} \to \mathcal{H}$ by the formula:

$$V_i\xi = U(\xi \otimes \eta_i)$$

for $\xi \in \mathcal{H}$. Then clearly $V_i$'s are isometries with orthogonal range projections. Note that $\sum_{i=1}^{d} V_iV_i^* = P$.

Let $\xi \in \mathcal{H}$ be given. Fix $A \in \mathcal{K}(\mathcal{H})$. Calculate as follows to observe that,

$$\begin{aligned} \alpha(A)V_i\xi &= U(A \otimes 1)U^*U(\xi \otimes \eta_i) \\ &= U(A\xi \otimes \eta_i) \\ &= V_iA\xi. \end{aligned}$$

Thus $\alpha(A)V_i = V_iA$. Post multiply by $V_i^*$ and add to see that for every finite $n \leq d$,

$$\alpha(A)(\sum_{i=1}^{n} V_iV_i^*) = \sum_{i=1}^{n} V_iAV_i^*.$$

Letting $n \to d$, we conclude that $\alpha(A) = \beta(A)$ where $\beta(A) := \sum_{i=1}^{d} V_iAV_i^*$ for every compact operator $A$.

Theorem 1.5 implies that $\beta$ is normal. As $\alpha$ is normal and $\mathcal{K}(\mathcal{H})$ is $\sigma$-weakly dense in $B(\mathcal{H})$, we have

$$\alpha(A) = \sum_{i=1}^{d} V_iAV_i^*$$

for $A \in B(\mathcal{H})$. This completes the proof.

## 3. The space of intertwiners

Theorem 1.5 and Theorem 2.3 gives a complete characterisation of a single normal endomorphism. However the isometries appearing in Theorem 2.3 is far from unique. In other words, Theorem 2.3 is basis/coordinate dependent. A coordinate free way of describing a single endomorphism is achieved through the notion of the space of intertwiners, a notion which is extremely fundamental in Arveson's programme of $E_0$-semigroups.

**Definition 3.1.** *Suppose $\alpha : B(\mathcal{H}) \to B(\mathcal{H})$ is a normal endomorphism. Let*

$$E_\alpha := \{T \in B(\mathcal{H}) : \alpha(A)T = TA \text{ for all } A \in B(\mathcal{H})\}.$$

*The space $E_\alpha$ is called the intertwining space of $\alpha$.*

Let $\alpha$ be a normal endomorphism of $B(\mathcal{H})$ and let $E = E_\alpha$ be the intertwining space of $\alpha$. Note that $E$ consists precisely of the intertwiners between the identity representation and $\alpha$. Observe the following properties of $E$.

(1) The set $E$ is a norm closed subspace of $B(\mathcal{H})$.

(2) Let $T, S \in E$ be given. Calculate as follows to observe that for $A \in B(\mathcal{H})$,

$$\begin{aligned} T^*SA &= T^*\alpha(A)S \\ &= AT^*S. \end{aligned}$$

Thus, $T^*S$ lies in the commutant of $B(\mathcal{H})$. Consequently, $T^*S$ is a scalar which we denote by $\langle S|T\rangle$.

(3) Note that the map $E \times E \ni (S,T) \to \langle S|T\rangle = T^*S \in \mathbb{C}$ is an inner product on $E$. Moreover the norm induced by the inner product coincides with the operator norm. Since $E$ is norm closed, it follows that $E$ is a Hilbert space with respect to the inner product $\langle\ |\ \rangle$. We always consider $E$ as a Hilbert space with respect to this inner product.

(4) Let $\{V_i\}_{i=1}^d$ be as in Theorem 2.3. It is clear that $V_i \in E$ and $\{V_i\}_{i=1}^d$ is an orthonormal set of $E$. We claim that $\{V_i\}_{i=1}^d$ is an orthonormal basis for $E$. Suppose $R \in E$ is such that $\langle R|V_i\rangle = 0$ for every $i$. Then $R^*V_i = 0$. Consequently, for $A \in B(\mathcal{H})$, $R^*\alpha(A) = \sum_{i=1}^d R^*V_iAV_i^* = 0$. But $R \in E$. Hence $AR^* = R^*\alpha(A) = 0$ for $A \in B(\mathcal{H})$. This implies that $R^* = 0$ and hence $R = 0$. This proves our claim. A consequence of this fact is that $E$ is a separable Hilbert space.

(5) We write $E\mathcal{H}$ for the closed linear span of $\{T\xi : T \in E, \xi \in \mathcal{H}\}$. Since $\{V_i\}_{i=1}^d$ is an orthonormal basis for $E$, it follows that the projection corresponding to $E\mathcal{H}$ is $\sum_{i=1}^d V_iV_i^* = \alpha(1)$. Note that if $\{W_i\}$ is another orthonormal basis of $E$, then $E\mathcal{H}$ is the closed linear span of $\{W_i\xi : i = 1, 2, \cdots, d, \xi \in \mathcal{H}\}$. Hence the projection corresponding to $E\mathcal{H}$ is $\sum_{i=1}^d W_iW_i^*$. As a consequence, $\alpha(1) = \sum_{i=1}^d W_iW_i^*$ for any orthonormal basis of $E$. Note that $\alpha$ is unital if and only if $E\mathcal{H} = \mathcal{H}$.

(6) Let $\{W_i\}_{i=1}^d$ be any orthonormal basis of $E$. We claim that $\alpha(A) = \sum_{i=1}^d W_iAW_i^*$. Fix $A \in B(\mathcal{H})$. Note that $\alpha(A)W_i = W_iA$. Post multiply by $W_i^*$ and add up to

$d$ to see that

$$\alpha(A) = \alpha(A)\alpha(1) = \alpha(A)(\sum_{i=1}^{d} W_iW_i^*) = \sum_{i=1}^{d} W_iAW_i^*.$$

This proves our claim.

The following is a corollary of our discussions.

**Corollary 3.2.** *Let $\alpha$ and $\beta$ be normal endomorphisms of $B(\mathcal{H})$. Then $\alpha = \beta$ if and only if $E_\alpha = E_\beta$.*

**Definition 3.3.** *Let $E \subset B(\mathcal{H})$ be a norm closed subspace. We call $E$ an intertwining space if*

1. *the space $E$ is separable as a Banach space, and*
2. *for every $S, T \in E$, $\langle S|T\rangle := T^*S$ is a scalar.*

Let $E$ be an intertwining space. Then $\langle\ |\ \rangle$ is an inner product on $E$ and the norm inherited via this inner product coincides with the operator norm. Consequently, $E$ is a separable Hilbert space. We always view interwining spaces as Hilbert spaces. The following proposition justifies our terminology.

**Proposition 3.4.** *Let $E \subset B(\mathcal{H})$ be an intertwining space. Then there exists a unique normal endomorphism $\alpha$ of $B(\mathcal{H})$ such that $E = E_\alpha$.*

*Proof.* Let $\{V_i\}_{i=1}^{d}$ be an orthonormal basis of $E$. Define $\alpha(A) = \sum_{i=1}^{d} V_iAV_i^*$. From the discussions preceding Corollary 3.2, it follows that $E_\alpha$ is the closed linear span, the closure is taken in the operator norm topology, of $\{V_i\}_{i=1}^{d}$. Hence $E_\alpha = E$. Uniqueness follows from Corollary 3.2.

**Remark 3.5.** *Summarising our discussions so far, we conclude that*

$$\alpha \to E_\alpha$$

*sets up a one-one correspondence between the set of normal endomorphisms of $B(\mathcal{H})$ and the set of intetwining spaces. So in principle, anything we say about the endomorphism $\alpha$ says something about the intertwining space $E_\alpha$ and vice versa.*

A nice application of the correspondence between endomorphisms and intertwining spaces is the following.

**Proposition 3.6.** *Let $\alpha$ and $\beta$ be normal endomorphisms on $B(\mathcal{H})$ and $B(\mathcal{K})$ respectively. Then there exists a unique normal endomorphism, denoted $\alpha \otimes \beta$, on $B(\mathcal{H} \otimes \mathcal{K})$ such that*

$$\alpha \otimes \beta(A \otimes B) = \alpha(A) \otimes \beta(B)$$

*for* $A \in B(\mathcal{H})$ *and* $B \in B(\mathcal{K})$.

*Proof.* Uniqueness follows from the fact the linear span of $\{A \otimes B : A \in B(\mathcal{H}), B \in B(\mathcal{K})\}$ is $\sigma$-weakly dense in $B(\mathcal{H} \otimes \mathcal{K})$. Let $E$ be the intertwining space of $\alpha$ and $F$ be the intertwining space of $\beta$. Define $E \otimes F$ to be the closure in the operator norm topology of the linear span of $\{S \otimes T : S \in E, T \in F\}$. It is clear that $E \otimes F$ is an intertwining space. As a Hilbert space, $E \otimes F$ is the tensor product of $E$ and $F$.

Denote the normal endomorphism corresponding to $E \otimes F$ by $\alpha \otimes \beta$. We claim that

$$\alpha \otimes \beta(A \otimes B) = \alpha(A) \otimes \beta(B)$$

for $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. Note that the projection $\alpha(1) \otimes \beta(1)$ acts as identity operator on $(E \otimes F)(\mathcal{H} \otimes \mathcal{K})$. Thus $\alpha \otimes \beta(1) \leq \alpha(1) \otimes \beta(1)$.

Fix $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. Let $\{V_i\}_{i=1}^{r}$ and $\{W_j\}_{j=1}^{s}$ be orthonormal bases for $E$ and $F$ respectively. Post multiply the equation

$$\alpha \otimes \beta(A \otimes B)(V_i \otimes W_j) = (V_i \otimes W_j)(A \otimes B)$$

by $V_i^* \otimes 1$ and add up to $r$ to obtain

$$\alpha \otimes \beta(A \otimes B)(\alpha(1) \otimes W_j) = \alpha(A) \otimes W_j B$$

Post multiply the above equation by $1 \otimes W_j^*$ and add to arrive at the equation

$$\alpha \otimes \beta(A \otimes B)(\alpha(1) \otimes \beta(1)) = \alpha(A) \otimes \beta(B).$$

The conclusion follows since $\alpha \otimes \beta(1) \leq \alpha(1) \otimes \beta(1)$. This completes the proof. □

Let $\alpha$ and $\beta$ be unital normal endomorphisms of $B(\mathcal{H})$. We denote the space of intertwiners between $\alpha$ and $\beta$ by $L(\alpha, \beta)$, i.e.

$$L(\alpha, \beta) = \{T \in B(\mathcal{H}) : T\alpha(A) = \beta(A)T, \text{ for } A \in B(\mathcal{H})\}.$$

In particular, $L(\alpha, \alpha) = \alpha(B(\mathcal{H}))'$. Let $E$ and $F$ be the intertwining space of $\alpha$ and $\beta$ respectively. Let $C \in L(\alpha, \beta)$ be given.

(1) Observe that $CT \in F$ whenever $T \in E$.
(2) Define $\theta_C : E \to F$ by $\theta_C(T) = CT$. Then, it is clear that $\theta_C$ is a bounded linear operator from $E \to F$.

**Exercise 3.1.** *With the foregoing notation, the map* $L(\alpha, \beta) \ni C \to \theta_C \in B(E, F)$ *is an isometric isomorphism. Show that* $\alpha(B(\mathcal{H}))' \cong B(E)$ *as von Neumann algebras.*

*Hint:* The map $E \otimes \mathcal{H} \ni T \otimes \xi \to T\xi \in \mathcal{H}$ is a unitary.

## 4. $E_0$-SEMIGROUPS

In this section, we define the basic object of our study "*$E_0$-semigroups*" and we discuss the most basic example of the subject "*the CCR flows*". For the rest of these notes, unless otherwise mentioned, the letter $G$ stands for an arbitrary locally compact second countable Hausdorff topological group and the letter $P$ stands for a closed subsemigroup of $G$ containing the identity element $e$. We assume that the interior of $P$, denoted $\Omega$, is dense in $P$.

Although we do not discuss any particular example of a semigroup, it is better to have a few examples to keep in mind and they are given below.

(1) The classical case is when $G = \mathbb{R}$ and $P = [0, \infty)$.
(2) Set $G = \mathbb{R}^d$ and let $P$ be a closed convex in $\mathbb{R}^d$ such that $P - P = \mathbb{R}^d$.
(3) $ax + b$ **semigrop:** Let
$$G := \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a > 0, b \in \mathbb{R} \right\}$$
and let
$$P := \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \geq 1, b \geq 0 \right\}.$$
(4) **Heisenberg beak:** Let
$$G := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$
and let
$$P := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \geq 0 \right\}.$$
(5) Let $G = \mathbb{Z}^d$ and $P$ be a finitely generated subsemigroup of $\mathbb{Z}^d$. We can assume that $P - P = \mathbb{Z}^d$.

**Definition 4.1.** *Let $\mathcal{H}$ be a separable Hilbert space. By an $E_0$-semigroup, over $P$, on $B(\mathcal{H})$, we mean a family $\alpha := \{\alpha_x\}_{x \in P}$ such that*

(a) *for $x \in P$, $\alpha_x$ is a normal endomorphism of $B(\mathcal{H})$,*
(b) *for $x \in P$, $\alpha_x$ is unital, i.e. $\alpha_x(1) = 1$, and*
(c) *for $T \in \mathcal{L}^1(\mathcal{H})$ and $A \in B(\mathcal{H})$, the map $P \ni x \to Tr(\alpha_x(A)T) \in \mathbb{C}$ is continuous.*

Noting the fact that endomorphisms are contractive, we see immediately that $(c)$ is equivalent to the following condition:

$(c)'$ for $A \in B(\mathcal{H})$, $\xi, \eta \in \mathcal{H}$, the map $P \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$ is continuous.

Since the semigroup $P$ will be fixed for the rest of this section, we simply call an *$E_0$-semigroup over $P$* an *$E_0$-semigroup.*

**Lemma 4.2.** *Let $\alpha := \{\alpha_x\}_{x \in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$. Then for $A \in B(\mathcal{H})$, the map $P \ni x \to \alpha_x(A) \in B(\mathcal{H})$ is continuous when $B(\mathcal{H})$ is given the strong operator topology.*

*Proof.* Let $U \in B(\mathcal{H})$ be unitary. Then $(c)'$ implies that the map $P \ni x \to \alpha_x(U) \in B(\mathcal{H})$ is continuous when $B(\mathcal{H})$ is given the weak operator topology. But $\{\alpha_x(U)\}_{x \in P}$ is a family of unitaries and the weak operator topology coincides with the strong operator one on the set of unitary operators. Thus the map $P \ni x \to \alpha_x(U) \in B(\mathcal{H})$ is continuous when $B(\mathcal{H})$ is given the strong operator topology. Now the proof is completed by appealing to the fact that every bounded operator can be written as a linear combination of four unitary operators. $\square$

**Definition 4.3.** *Let $\alpha := \{\alpha_x\}_{x \in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$. Suppose $U := \{U_x\}_{x \in P}$ is a strongly continuous family of unitaries. We say that $U$ is an $\alpha$-cocycle if*

$$U_x \alpha_x(U_y) = U_{xy}$$

*for $x, y \in P$.*

**Exercise 4.1.** *Let $\alpha := \{\alpha_x\}_{x \in P}$ be an $E_0$-semigroup and $U := \{U_x\}_{x \in P}$ be an $\alpha$-cocycle. Define for $x \in P$, $\beta_x : B(\mathcal{H}) \to B(\mathcal{H})$ by*

$$\beta_x(A) = U_x \alpha_x(A) U_x^*.$$

*Use Lemma 4.2 to prove that $\beta := \{\beta_x\}_{x \in P}$ is an $E_0$-semigroup.*

**Definition 4.4.** *Let $\alpha := \{\alpha_x\}_{x \in P}$ and $\beta := \{\beta_x\}_{x \in P}$ be $E_0$-semigroups on $B(\mathcal{H})$. We say $\beta$ is a cocycle perturbation of $\alpha$ and write $\beta \simeq \alpha$ if there exists an $\alpha$-cocycle $U := \{U_x\}_{x \in P}$ such that $\beta_x(.) = U_x \alpha_x(.) U_x^*$.*

Observe that $\simeq$ is an equivalence relation on the set of $E_0$-semigroups on $B(\mathcal{H})$. The equivalence relation $\simeq$ is the most basic equivalence relation in the subject. If we have two $E_0$-semigroups acting on different Hilbert spaces, first we bring both to a common Hilbert space and compare whether one is a cocycle perturbation of the other.

Let us be more precise. For a unitary $U : \mathcal{H} \to \mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces, we denote the map $B(\mathcal{H}) \ni T \to UTU^* \in B(\mathcal{K})$ by $Ad(U)$. Let $\alpha$ and $\beta$ be $E_0$-semigroups on $B(\mathcal{H})$ and $B(\mathcal{K})$. We say $\alpha$ is *conjugate* to $\beta$ if there exists a unitary $U : \mathcal{H} \to \mathcal{K}$ such that $\beta_x = Ad(U) \circ \alpha_x \circ Ad(U)^*$. We say $\alpha$ and $\beta$ are *cocycle conjugate* if a conjugate of $\alpha$ is a cocycle perturbation of $\beta$.

**Example 4.5.** *Let $U := \{U_x\}_{x\in P}$ be a family of unitaries on $\mathcal{H}$ which is strongly continuous. Assume that $U_xU_y = U_{xy}$. Define $\alpha_x : B(\mathcal{H}) \to B(\mathcal{H})$ by*

$$\alpha_x(A) = U_xAU_x^*.$$

*Then $\alpha := \{\alpha_x\}_{x\in P}$ is an $E_0$-semigroup. Note that each $\alpha_x$ is an automorphism. Such an $E_0$-semigroup is called an automorphism group. Clearly $\alpha$ is cocycle conjugate to the "identity $E_0$-semigroup", i.e. the $E_0$-semigroup for which each of its endomorphism is the identity map.*

When $P = [0,\infty)$, Wigner's theorem asserts that every automorphism group is one of the above type. We will have more to say on this later. The above example can be generalised as follows. Let $\mathbb{T}$ be the unit circle. Suppose $\omega : G \times G \to \mathbb{T}$ is a measurable map. We say $\omega$ is a multiplier, or a Borel multiplier, on $G$, if

$$\omega(x,y)\omega(xy,z) = \omega(x,yz)\omega(y,z)$$

for $x,y,z \in G$.

By an *$\omega$-projective unitary representation of* $G$ on a Hilbert space $\mathcal{H}$, we mean a family of unitaries $U := \{U_x\}_{x\in G}$ on $\mathcal{H}$ such that

$$U_xU_y = \omega(x,y)U_{xy}$$

for $x,y \in G$ and $U$ is weakly measurable, i.e. for $\xi,\eta \in \mathcal{H}$, the map $G \ni x \to \langle U_x\xi|\eta\rangle \in \mathbb{C}$ is measurable.

**Example 4.6.** *Let $\omega$ be a Borel multiplier on $G$ and $U := \{U_x\}_{x\in P}$ be a $\omega$-projective unitary representation of $G$ on $\mathcal{H}$. Define for $x \in P$, $\alpha_x : B(\mathcal{H}) \to B(\mathcal{H})$ by*

$$\alpha_x(A) = U_xAU_x^*.$$

*It is clear that $\alpha := \{\alpha_x\}_{x\in P}$ is a semigroup of unital normal endomorphisms. The weak measurability of $U$ implies that for $A \in B(\mathcal{H})$ and $\xi,\eta \in \mathcal{H}$, the map $P \ni x \to \langle\alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$ is measurable. This is sufficient to ensure that $\alpha$ is an $E_0$-semigroup (see Section 8).*

Let $\omega$ be a Borel multiplier on $G$. The following is the "left" regular $\omega$-projective representation. Let $\mathcal{H} = L^2(G)$. Define for $x \in G$, the unitary operator $U_x$ on $\mathcal{H}$ by

$$U_xf(y) = \omega(x,x^{-1}y)f(x^{-1}y)$$

for $f \in L^2(G)$.

## 5. Fock spaces

We proceed towards defining the basic examples of the theory : **CCR and CAR** flows. First, we review the symmetric and anti-symmetric Fock spaces first. Let $\mathcal{H}$ be a separable Hilbert space fixed for this section. For $n \geq 1$, set

$$\mathcal{H}^{\otimes n} := \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n\ times}.$$

We set $\mathcal{H}^{\otimes 0} := \mathbb{C}$. For $n \geq 1$, let $S_n$ be the symmetric group on $\{1, 2, \cdots, n\}$. For $\sigma \in S_n$, we denote the sign of $\sigma$ by $\epsilon(\sigma)$. Thus $\epsilon(\sigma) = 1$ if $\sigma$ is a product of even number of transpositions and $\epsilon(\sigma) = -1$ if it is a product of odd number of transpositions. For $\sigma \in S_n$, let $U_\sigma$ be the unitary on $\mathcal{H}^{\otimes n}$ defined by the equation

$$U_\sigma(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) := \xi_{\sigma^{-1}(1)} \otimes \xi_{\sigma^{-1}(2)} \otimes \cdots \otimes \xi_{\sigma^{-1}(n)}$$

for $\xi_1, \xi_2, \cdots, \xi_n \in \mathcal{H}$.

The space of symmetric tensors, denoted $\mathcal{H}_s^{\otimes n}$, is defined as

$$\mathcal{H}_s^{\otimes n} := \{\xi \in \mathcal{H}^{\otimes n} : U_\sigma(\xi) = \xi\}.$$

For $n = 0$, we set $\mathcal{H}_s^0 = \mathbb{C}$. The *symmetric Fock space*, denoted $\Gamma_s(\mathcal{H})$, is defined as

$$\Gamma_s(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_s^{\otimes n}.$$

The symmetric Fock space is also called the space of *Bosons* by physicists. The vector $1 \oplus 0 \oplus 0 \oplus \cdots \in \Gamma_s(\mathcal{H})$ is called *the vacuum vector* or *the vacuum state*. Usually it is denoted by the letter $\Omega$. But since for us, $\Omega$ stands for the interior of the semigroup $P$, we do not assign a special symbol for the vacuum state.

For $n \geq 1$, let $\wedge^n \mathcal{H}$ be defined by

$$\wedge^n \mathcal{H} = \{\xi \in \mathcal{H}^{\otimes n} : U_\sigma \xi = \epsilon(\sigma)\xi\}.$$

For $n = 0$, we set $\wedge^0 \mathcal{H} = \mathbb{C}$. We call $\wedge^n \mathcal{H}$ as the space of anti-symmetric tensors in $\mathcal{H}^{\otimes n}$. The *anti-symmetric Fock space*, denoted $\Gamma_a(\mathcal{H})$, is defined as

$$\Gamma_a(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \wedge^n \mathcal{H}.$$

Physicists call the anti-symmetric Fock space as the space of *Fermions*. Again the vector $1 \oplus 0 \oplus 0 \oplus \cdots$ is called the *vacuum vector* or the *vacuum state*.

Let us first discuss the symmetric Fock space is some detail. We begin with a little lemma. For $\xi \in \mathcal{H}$ and $n \geq 1$, set $\xi^{\otimes n} := \underbrace{\xi \otimes \xi \otimes \cdots \otimes \xi}_{n\ times}$. For $n = 0$, we set $\xi^{\otimes 0} = 1$.

**Lemma 5.1.** *For every $n \geq 1$, the set $\{\xi^{\otimes n} : \xi \in \mathcal{H}\}$ is total in $\mathcal{H}_s^{\otimes n}$.*

*Proof.* The statement is clearly true for $n=1$. Assume $n\geq 2$. Let $P$ be the projection of $\mathcal{H}^{\otimes n}$ onto $\mathcal{H}_s^{\otimes n}$. Then $P=\dfrac{1}{n!}\displaystyle\sum_{\sigma\in S_n}U_\sigma$. Denote the closed linear span of $\{\xi^{\otimes n}:\xi\in\mathcal{H}\}$ by $D$. Let $\xi_1,\xi_2,\cdots,\xi_n\in\mathcal{H}$ be given. Define $f:\mathbb{R}^n\to D$ by

$$f(t_1,t_2,\cdots,t_n)=(t_1\xi_1+t_2\xi_2+\cdots+t_n\xi_n)^{\otimes n}.$$

Note that $f$ is a polynomial taking values in $D$. Consequently, all its coefficients are in $D$. Note that the coefficient of $t_1t_2\cdots t_n$ is $n!$ times $P(\xi_1\otimes\xi_2\otimes\cdots\xi_n)$. Now the lemma follows. $\square$

For $\xi\in\mathcal{H}$, let

$$e(\xi):=\sum_{n=0}^{\infty}\frac{\xi^{\otimes n}}{\sqrt{n!}}.$$

Note that $e(\xi)\in\Gamma_s(\mathcal{H})$. The set $\{e(\xi):\xi\in\mathcal{H}\}$ is called the set of *exponential vectors.* For $\xi,\eta\in\mathcal{H}$,

$$\langle e(\xi)|e(\eta)\rangle=e^{\langle\xi|\eta\rangle}.$$

**Proposition 5.2.** *The set of exponential vectors is total in $\Gamma_s(\mathcal{H})$.*

*Proof.* Let $\xi\in\mathcal{H}$ be given. Denote the closed linear span of exponential vectors by $D$. Define $f:\mathbb{R}\to D$ by $f(t)=e(t\xi)$. Note that $f$ is analytic taking values in $D$. Hence its coefficients are in $D$. Note that the coefficient of $t^n$ is $\frac{\xi^{\otimes n}}{\sqrt{n!}}$. An appeal to Lemma 5.1 completes the proof. $\square$

**Remark 5.3.** *We will repeatedly make use of the following. Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $S_1$ and $S_2$ are total subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let $\phi:S_1\to S_2$ be a map such that $\langle\phi(x)|\phi(y)\rangle=\langle x|y\rangle$ for $x,y\in S_1$. Then there exists a unique isometry $V:\mathcal{H}_1\to\mathcal{H}_2$ which extends $\phi$. Moreover if $\phi$ is a bijection, the isometry $V$ is a unitary.*

The following statements follows directly from the previous remark and the totality of exponential vectors.

(1) Let $\xi\in\mathcal{H}$ be given. Then there exists a unique unitary operator on $\Gamma_s(\mathcal{H})$, denoted $W(\xi)$, such that

$$W(\xi)e(\eta)=e^{-\frac{||\xi||^2}{2}-\langle\eta|\xi\rangle}e(\eta+\xi).$$

The operators $\{W(\xi):\xi\in\mathcal{H}\}$ are called the set of *Weyl operators.* Moreover the Weyl operators satisfy the following relations called *the canonical commutation relations* abbreviated as CCR. For $\xi,\eta\in\mathcal{H}$,

$$W(\xi)W(\eta)=e^{iIm\langle\xi|\eta\rangle}W(\xi+\eta)$$

where $Im(\langle\xi|\eta\rangle)$ denotes the imaginary part of $\langle\xi|\eta\rangle$.

(2) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $U : \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary operator. Then there exists a unique unitary operator, called the *second quantisation* of $U$, denoted $\Gamma(U)$ such that for $\xi \in \mathcal{H}_1$, $\Gamma(U)e(\xi) = e(U\xi)$. Observe that for $\xi \in \mathcal{H}_1$ and a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$, $\Gamma(U)W(\xi)\Gamma(U)^* = W(U\xi)$.

(3) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then there exists a unique unitary operator from $\Gamma_s(\mathcal{H}_1 \oplus \mathcal{H}_2) \to \Gamma_s(\mathcal{H}_1) \otimes \mathcal{H}_s(\mathcal{H}_2)$ which maps $e(\xi_1 \oplus \xi_2)$ to $e(\xi_1) \otimes e(\xi_2)$. We always identify $\Gamma_s(\mathcal{H}_1 \oplus \mathcal{H}_2)$ with $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$ via this identification. Under this identification, note that for $\xi_1 \in \mathcal{H}_1$ and $\xi_2 \in \mathcal{H}_2$,
$$W(\xi_1 \oplus \xi_2) = W(\xi_1) \otimes W(\xi_2).$$

Note that the CCR relations imply that the linear span of $\{W(\xi) : \xi \in \mathcal{H}\}$ is a unital $*$-subalgebra of $B(\Gamma_s(\mathcal{H}))$. The fundamental fact that we do not prove, but we need in the sequel, is stated below. The reader is referred to Prop. 20.9 of [23] for a proof.

**Theorem 5.4.** *The linear span of $\{W(\xi) : \xi \in \mathcal{H}\}$ is a unital $*$-subalgebra of $B(\Gamma_s(\mathcal{H}))$ and it is strongly dense in $B(\Gamma_s(\mathcal{H}))$.*

**Exercise 5.1.** *Let $U : \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary operator. Show that $\Gamma(U)\xi^{\otimes n} = (U\xi)^{\otimes n}$.*

*Hint:* Use the ideas of the proof of 5.2.

**Exercise 5.2.** *Let $A$ be a contraction on $\mathcal{H}$, i.e. $||A|| \leq 1$. Show that there exists a unique operator on $\Gamma_s(\mathcal{H})$ such that $\Gamma(A)e(\xi) = e(A\xi)$.*

Let us review the facts about the anti-symmetric Fock space. Proofs are mostly omitted as it involves repeating the "Exterior algebra" construction. For $\xi \in \mathcal{H}^{\otimes n}$, let
$$Alt(\xi) := \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) U_\sigma(\xi).$$
Note that $Alt$ is the orthogonal projection of $\mathcal{H}^{\otimes n}$ onto $\wedge^n \mathcal{H}$. For $\xi \in \wedge^m \mathcal{H}$ and $\eta \in \wedge^n \mathcal{H}$, define
$$\xi \wedge \eta := \frac{(m+n)!}{m!n!} Alt(\xi \otimes \eta).$$
Then $\xi \wedge \eta = (-1)^{mn} \eta \wedge \xi$. Moreover, $\wedge$ defines an associative multiplication on the algebraic direct sum $\bigoplus_{alg} \wedge^n \mathcal{H}$. Note that $\{\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n : \xi_i \in \mathcal{H}\}$ is total in $\wedge^n \mathcal{H}$. We have the following formula for the inner product.

Let $\xi_1, \xi_2, \cdots, \xi_n$ and $\eta_1, \eta_2, \cdots, \eta_n \in \mathcal{H}$ be given. Then
$$\langle \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n | \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_n \rangle = \det(\langle \xi_i | \eta_j \rangle).$$

Let $D$ be the linear span of $\{\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n\}$ where $\xi_i$'s vary over $\mathcal{H}$ and $n$ varies over $\{0, 1, 2, \cdots\}$. An empty product, i.e. when $n = 0$, is interpreted as the element 1 in $\wedge^0\mathcal{H} = \mathbb{C}$. Let us define the *creation* and *annihilation* operators on $D$. For $\xi \in \mathcal{H}$, let $a(\xi)$ and $a(\xi)^*$ be defined on $D$ by the following formulas:

$$a(\xi)(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n) := \xi \wedge \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n$$
$$a(\xi)^*(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n) := \sum_{k=1}^{n}(-1)^{k-1}\langle \xi_k|\xi\rangle \xi_1 \wedge \xi_2 \wedge \cdots \wedge \widehat{\xi_k} \wedge \cdots \wedge \xi_n$$

where the "hat" symbol indicates the omission of the vector. The operators $\{a(\xi) : \xi \in \mathcal{H}\}$ are called *creation* operators and the operators $\{a(\xi)^* : \xi \in \mathcal{H}\}$ are called *annihilation* operators. It is a good exercise in multilinear algebra to prove that the creation and annihilation operators are well defined on $D$. The choice of the notation for annihilation operators is justified in the next Lemma.

**Lemma 5.5.** *Let $\xi \in \mathcal{H}$ be given. Then for $u, v \in D$, $\langle a(\xi)u|v\rangle = \langle u|a(\xi)^*v\rangle$.*

*Proof.* Let $\xi_1, \xi_2, \cdots, \xi_n \in \mathcal{H}$ and $\eta_1, \eta_2, \cdots, \eta_{n+1} \in \mathcal{H}$ be given. It is sufficient to show that

$$\langle a(\xi)(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n)|\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_{n+1}\rangle = \langle \xi_1 \wedge \xi_2 \wedge \cdots \xi_n|a(\xi)^*(\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_{n+1})\rangle.$$

Note that the RHS is $\sum_{k=1}^{n+1}(-1)^{k-1}\langle \xi|\eta_k\rangle\langle \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n|\eta_1 \wedge \cdots \wedge \widehat{\eta_k} \wedge \cdots \wedge \eta_{n+1}\rangle$ which is the expression we obtain when we expand the determinant of $\langle \widetilde{\xi_i}|\eta_j\rangle$ along the first row. Here $\widetilde{\xi_0} = \xi$ and $\widetilde{\xi_i} = \xi_{i-1}$. But the value of the latter determinant is exactly $\langle a(\xi)(\xi_1 \wedge \cdots \wedge \xi_n|\eta_1 \wedge \cdots \wedge \eta_{n+1}\rangle$. This completes the proof.

A direct calculation reveals that, on $D$, the creation and annihilation operators satisfy the following *canonical anti-commutation* relations abbreviated as CAR. For $\xi, \eta \in \mathcal{H}$,

$$a(\xi)a(\eta) + a(\eta)a(\xi) = 0$$
$$a(\xi)a(\eta)^* + a(\eta)^*a(\xi) = \langle \xi|\eta\rangle$$

Clearly the map $\mathcal{H} \ni \xi \to a(\xi)$ is linear. The next proposition shows that creation and annihilation operators extend to bounded operators on $\Gamma_a(\mathcal{H})$. The reason is simple. For a unit vector $\xi$, the CAR relations imply that "$a(\xi)a(\xi)^* \leq 1$".

**Proposition 5.6.** *For $\xi \in \mathcal{H}$ and $u \in D$, $||a(\xi)u||^2 \leq ||\xi||^2||u||^2$. The same estimate holds for $a(\xi)^*$. We denote the unique extension of $a(\xi)$ and $a(\xi)^*$ to $\Gamma_a(\mathcal{H})$ again by $a(\xi)$ and $a(\xi)^*$ respectively.*

*Proof.* Let $\xi \in \mathcal{H}$ and $u \in D$ be given. Calculate as follows to observe that

$$\begin{aligned}
||a(\xi)u||^2 &\leq ||a(\xi)u||^2 + ||a(\xi)^*u||^2 \\
&= \langle a(\xi)u|a(\xi)u\rangle + \langle a(\xi)^*u|a(\xi)^*u\rangle \\
&= \langle (a(\xi)^*a(\xi) + a(\xi)a(\xi)^*)u|u\rangle \\
&= \langle ||\xi||^2u|u\rangle \\
&\leq ||\xi||^2||u||^2.
\end{aligned}$$

This completes the proof. □

The creation and annihilation operators, now defined on $\Gamma_a(\mathcal{H})$, satisfy the CAR relations. Let $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$. Let $\mathcal{A}$ be the linear span of

$$\{a(\xi_1)a(\xi_2)\cdots a(\xi_m)a(\eta_1)^*a(\eta_2)^*\cdots a(\eta_n)^* : \xi_1, \xi_2, \cdots, \xi_m, \eta_1, \eta_2 \cdots, \eta_n \in \mathcal{H}, m, n \in \mathbb{N}_0\}.$$

An empty product is interpreted as the identity operator.

**Proposition 5.7.** *With the foregoing notation, the vector space $\mathcal{A}$ is a unital $*$-subalgebra of $B(\Gamma_a(\mathcal{H}))$. Moreover $\mathcal{A}$ is strongly dense in $B(\Gamma_a(\mathcal{H}))$*

*Proof.* The fact that $\mathcal{A}$ is a unital $*$-subalgebra follows from the CAR relations. It suffices to show that the commutant of $\mathcal{A}$ is $\mathbb{C}$. Let $v$ be the vacuum vector of $\Gamma_a(\mathcal{H})$. We claim that $\displaystyle\bigcap_{\xi \in \mathcal{H}} ker(a(\xi)^*) = \mathbb{C}v$. By definition, $a(\xi)^*v = 0$ for every $\xi \in \mathcal{H}$. Consider a vector $u \in \Gamma_a(\mathcal{H})$ such that $a(\xi)^*u = 0$. Note that for $k \in \mathbb{N}_0$ and $\xi_1, \cdots, \xi_{k+1} \in \mathcal{H}$,

$$\langle u|\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{k+1}\rangle = \langle a(\xi_1)^*u|\xi_2 \wedge \xi_3 \wedge \cdots \wedge \xi_{k+1}\rangle = 0.$$

Thus $u$ is orthogonal to $\wedge^n\mathcal{H}$ for $n \geq 1$. This implies that $u$ is a scalar multiple of $v$. This proves our claim.

Let $T \in \mathcal{A}'$ be given. Then $T$ leaves $\displaystyle\bigcap_{\xi \in \mathcal{H}} Ker(a(\xi)^*)$ invariant. Let $\lambda \in \mathbb{C}$ be such that $Tv = \lambda v$. Calculate as follows to observe that for $\xi_1, \xi_2 \cdots \xi_n \in \mathcal{H}$,

$$\begin{aligned}
T(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n) &= Ta(\xi_1)a(\xi_2)\cdots a(\xi_n)v \\
&= a(\xi_1)a(\xi_2)\cdots a(\xi_n)Tv \\
&= \lambda a(\xi_1)a(\xi_2)\cdots a(\xi_n)v \\
&= \lambda(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n).
\end{aligned}$$

Thus $T = \lambda$. This proves that $\mathcal{A}' = \mathbb{C}$ and hence the proof. □

**Second quantization:** Let $A$ be a contraction on $\mathcal{H}$. Denote the operator on $\mathcal{H}^{\otimes n}$, whose action is given below, by $A^{\otimes n}$:

$$A^{\otimes n}(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = A\xi_1 \otimes A\xi_2 \otimes \cdots \otimes A\xi_n.$$

Note that $\bigoplus_{n=0}^{\infty} A^{\otimes n}$ is a bounded operator on the full Fock space $\Gamma_f(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$. It leaves the the anti-symmetric Fock space (also the symmetric Fock space) invariant. We denote the restriction of it to the anti-symmetric Fock space by $\Gamma(A)$. We call $\Gamma(A)$ the *second quantization* of $A$. Observe that $\Gamma(A)(\xi_1 \wedge \xi_2 \wedge \cdots \xi_n) = A\xi_1 \wedge A\xi_2 \wedge \cdots \wedge A\xi_n$. Note that $\Gamma(A)$ is a unitary (isometry) if $A$ is unitary (isometry).

**Tensor product of anti-symmetric Fock spaces:** Just like the symmetric Fock space, the anti-symmetric Fock space of direct sum of two Hilbert spaces is the tensor product of the corresponding anti-symmetric Fock spaces. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. An application of Remark 5.3 immediately yields a unitary from $\Gamma_a(\mathcal{H}_1) \otimes \Gamma_a(\mathcal{H}_2) \to \Gamma_a(\mathcal{H}_1 \oplus \mathcal{H}_2)$ taking $(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_m) \otimes (\eta_1 \wedge \eta_2 \wedge \cdots \eta_n)$ to $\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_m \wedge \eta_1 \wedge \eta_2 \wedge \cdots \eta_n$. We always identify $\Gamma_a(\mathcal{H}_1 \oplus \mathcal{H}_2)$ with $\Gamma_a(\mathcal{H}_1) \otimes \Gamma_a(\mathcal{H}_2)$ via this unitary. Note that under this identification, we have the equality

$$a(\xi \oplus \eta) = a(\xi) \otimes 1 + \Gamma(-1) \otimes a(\eta).$$

## 6. CCR and CAR flows

In this section, we define the basic examples in the theory which are CCR and CAR flows.

**Proposition 6.1.** *Let $\mathcal{H}$ be a Hilbert space and $V$ be an isometry on $\mathcal{H}$. Then there exists a unique unital normal $*$-endomorphism $\alpha$ on $B(\Gamma_s(\mathcal{H}))$ such that*

$$\alpha(W(\xi)) = W(V\xi)$$

*for $\xi \in \mathcal{H}$.*

*Proof.* Uniqueness follows from the fact that the linear span of $\{W(\xi) : \xi \in \mathcal{H}\}$ is $\sigma$-weakly dense in $B(\Gamma_s(\mathcal{H}))$. The endomorphism $\alpha$ is described concretely as follows. Let $A \in B(\Gamma_s(\mathcal{H}))$ be given. Consider the operator $1 \otimes A$ on $\Gamma_s(Ker(V^*)) \otimes \Gamma(\mathcal{H})$. The isometry $V$ identifies $\mathcal{H}$ with $Ran(V)$. The operator $\alpha(A)$ is defined as the operator on $\Gamma_s(\mathcal{H})$ obtained from $1 \otimes A$ after we make the following identifications.

$$\Gamma_s(Ker(V^*)) \otimes \Gamma_s(\mathcal{H}) \simeq \Gamma_s(Ker(V^*)) \otimes \Gamma_s(RanV) \simeq \Gamma_s(Ker(V^*) \oplus RanV) \simeq \Gamma_s(\mathcal{H}).$$

Note that if we perform all the identifications, we obtain

$$1 \otimes W(\xi) \simeq W(0) \otimes W(V\xi) \simeq W(0 \oplus V\xi) \simeq W(V\xi).$$

Thus $\alpha(W(\xi)) = W(V\xi)$ for $\xi \in \mathcal{H}$. The normality of $\alpha$ follows from the fact the map

$$B(\Gamma_s(\mathcal{H})) \ni A \to 1 \otimes A \in B(\Gamma_s(Ker(V^*)) \otimes \Gamma_s(\mathcal{H}))$$

is continuous with respect to the $\sigma$-weak topology (this follows by applying a variant of 1.4). This completes the proof. $\square$

Thus to obtain a semigroup of endomorphisms, all we need is a semigroup of isometries indexed by $P$. This is made precise as follows.

**Definition 6.2.** *Let $V : P \to B(\mathcal{H})$ be a map. We denote the image of $x$, under $V$, by $V_x$. Then $V$ is called a strongly continuous isometric representation of $P$ if*

(1) *for $x \in P$, $V_x$ is an isometry,*
(2) *for $x, y \in P$, $V_x V_y = V_{xy}$, and*
(3) *for $\xi \in \mathcal{H}$, the map $P \ni x \to V_x\xi \in \mathcal{H}$ is continuous where $\mathcal{H}$ is given the norm topology.*

We only consider strongly continuous isometric representations and we simply call a strongly continuous isometric representation an isometric representation. The first example of an isometric representation is the "left" regular representation of $P$ on $L^2(P)$. One could generalise this slightly by considering $P$-invariant closed subsets and we can consider with multiplicity.

Let $A \subset G$ be a non-empty closed subset. We say $A$ is a *$P$-module* if $PA \subset A$. Let $A$ be a $P$-module and $\mathcal{K}$ be a Hilbert space of dimension $k$. Consider the Hilbert space $\mathcal{H} := L^2(A, \mathcal{K})$. For $x \in P$, define $V_x$ on $L^2(A, \mathcal{K})$ by the following formula:

$$V_x(f)(y) := \begin{cases} f(x^{-1}y) & \text{if } x^{-1}y \in A, \\ \\ 0 & \text{if } x^{-1}y \notin A \end{cases} \tag{6.3}$$

for $f \in L^2(A, \mathcal{K})$. Then $V := \{V_x\}_{x \in P}$ is a strongly continous isometric representation of $P$ on $L^2(A, \mathcal{K})$. We call $V$ *the isometric representation associated to the $P$-module $A$ of multiplicty $k$* and denote it by $V^{(A,k)}$. Note that $V_x$ is simply the compression of $\lambda_x$ on $L^2(A, \mathcal{K})$ where $\{\lambda_x\}_{x \in G}$ is the "left" regular representation of $G$ on $L^2(G, \mathcal{K})$.

We are now in a position to define our first example of an $E_0$-semigroup.

**Theorem 6.3.** *Let $V := \{V_x\}_{x \in P}$ be an isometric representation of $P$ on a Hilbert space $\mathcal{H}$. Then there exists a unique $E_0$-semigroup $\alpha^V := \{\alpha_x\}_{x \in P}$ on $B(\Gamma_s(\mathcal{H}))$ such that for $x \in P$ and $\xi \in \mathcal{H}$,*

$$\alpha_x(W(\xi)) = W(V_x\xi).$$

*Proof.* Uniqueness follows from the fact that linear span of $\{W(\xi) : \xi \in \mathcal{H}\}$, which we denote by $\mathcal{A}$, is $\sigma$-weakly dense in $B(\Gamma_s(\mathcal{H}))$. Fix $x \in P$. By Prop. 6.1, there exists a unique unital normal endomorphism, call it $\alpha_x$, such that $\alpha_x(W(\xi)) = W(V_x\xi)$ for

$\xi \in \mathcal{H}$. It is clear that $\alpha_x \circ \alpha_y = \alpha_{xy}$ on $\mathcal{A}$. The density of $\mathcal{A}$ and the normality of the endomorphisms involved imply that $\alpha_x \circ \alpha_y = \alpha_{xy}$ on $B(\Gamma_s(\mathcal{H}))$. Set $\alpha^V := \{\alpha_x\}_{x \in P}$.

By Exercise 6.1, it follows that for $x \in P$, $A \in \mathcal{A}$, and, $\xi, \eta \in \Gamma_s(\mathcal{H})$, the map $P \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$ is continuous. Let $A \in B(\Gamma_s(\mathcal{H}))$, $\xi, \eta \in \Gamma_s(\mathcal{H})$ be given. Note that $\mathcal{A}$ is dense in $B(\Gamma_s(\mathcal{H}))$ with respect to the strong operator topology. By Kaplansky density theorem, there exists a sequence $(A_n) \in \mathcal{A}$ with $||A_n|| \leq ||A||$ such that $A_n \to A$ in SOT. Fix $x \in P$. Since $\alpha_x$ is normal, it follows that $\alpha_x(A_n) \to \alpha_x(A)$ weakly. Thus $\langle \alpha_x(A_n)\xi|\eta\rangle \to \langle \alpha_x(A)\xi|\eta\rangle$. In other words, the sequence of continuous functions $P \ni x \to \langle \alpha_x(A_n)\xi|\eta\rangle \in \mathbb{C}$ converges pointwise to the map $P \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$. Consequently, the map $P \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$ is measurable. Now Theorem 7.1, due to Murugan, guarantees that $\alpha^V$ is an $E_0$-semigroup. This completes the proof. $\square$

**Exercise 6.1.** *Show that the map $\mathcal{H} \ni \xi \to W(\xi) \in B(\Gamma_s(\mathcal{H}))$ is continuous where $\mathcal{H}$ is given the norm topology and $B(\Gamma_s(\mathcal{H}))$ is given the weak operator topology. Use this to show the following. Keep the notation of Theorem 6.3. Prove that for $A \in \mathcal{A}$, $\xi, \eta \in \Gamma_s(\mathcal{H})$, the map $P \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$ is continuous.*

The $E_0$-semigroup $\alpha^V$ of Prop. 6.3 is called *the CCR flow associated to the isometric representation $V$*. Note that if each $V_x$ is unitary then $\alpha_x$ is an automorphism for every $x \in P$. In that case, $\alpha_x$ is $Ad(\Gamma(V_x))$. It is clear that if $V$ and $W$ are unitarily equivalent, then the associated CCR flows $\alpha^V$ and $\alpha^W$ are conjugate. Let us state two results regarding the classification of CCR flows. The first, a classical result, which kick started the subject, is due to Arveson and the second is due to Anbu Arjunan and the author.

For a $P$-module $A$ and $k \in \{1, 2, \cdots\} \cup \{\infty\}$, the CCR flow associated to the isometric representation $V^{(A,k)}$ is denoted $\alpha^{(A,k)}$ and we call $\alpha^{(A,k)}$ *the CCR flow associated to the module $A$ of multiplicity $k$*. Note that if $A$ and $B$ are $P$-modules and $A = Bz$ for some $z \in G$ then $\alpha^{(A,k)}$ and $\alpha^{(B,k)}$ are conjugate. For, the corresponding isometric representations are unitarily equivalent. We call a $P$-module *proper* if $A \neq G$.

**Exercise 6.2.** *Let $P = [0, \infty)$ and $A$ be a $P$-module. If $A \neq \mathbb{R}$ then there exists $x \in \mathbb{R}$ such that $A = [x, \infty)$. Thus, up to a translate, the only proper $[0, \infty)$ module is $[0, \infty)$.*

**Theorem 6.4** (Arveson)**.** *Let $k_1, k_2 \in \{1, 2, \cdots\} \cup \{\infty\}$ be given. The CCR flow $\alpha^{([0,\infty),k_1)}$ is cocycle conjugate to $\alpha^{([0,\infty),k_2)}$ if and only if $k_1 = k_2$.*

The 1-parameter CCR flow $\alpha^{([0,\infty),k)}$ is called *the CCR flow of index $k$* in the 1-parameter literature.

Assume $d \geq 2$. Let $P \subset \mathbb{R}^d$ be a closed convex cone which we assume is spanning, i.e. $P - P = \mathbb{R}^d$ and pointed, i.e. $P \cap -P = \{0\}$. Let $A_1$ and $A_2$ be proper $P$-modules and let $k_1, k_2 \in \{1, 2, \cdots\} \cup \{\infty\}$ be given. The main content of Anbu Arjunan's thesis and [4] is the following theorem.

**Theorem 6.5.** *Keep the foregoing notation. The following are equivalent.*

(1) *The CCR flow $\alpha^{(A_1,k_1)}$ is cocycle conjugate to $\alpha^{(A_2,k_2)}$.*
(2) *The modules $A_1$ and $A_2$ are translates of each other, i.e. there exists $z \in \mathbb{R}^d$ such that $A_1 + z = A_2$ and $k_1 = k_2$.*

The proof of the above theorem relies heavily on groupoids and in particular on the groupoid approach to the study of Wiener-Hopf operators initiated by Muhly and Renault in their seminal paper [18]. We must mention here that the above theorem was first proved by barehand methods for a few examples of $\mathbb{R}_+^2$-modules in [3].

Next we discuss the CAR flow associated to an isometric representation. Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{A}$ be the linear span of

$$\{a(\xi_1)\cdots a(\xi_m)a(\eta_1)^*\cdots a(\eta_n)^* : \xi_1, \xi_2, \cdots, \xi_m, \eta_1, \eta_2, \cdots, \eta_n \in \mathcal{H}, m, n \in \mathbb{N}_0\}.$$

**Proposition 6.6.** *Let $V$ be an isometry on a Hilbert space. Then there exists a unique unital endomorphism $\alpha$ on $B(\Gamma_a(\mathcal{H}))$ such that*

$$\alpha(a(\xi)) = a(V\xi)$$

*for $\xi \in \mathcal{H}$.*

*Proof.* The proof is similar to 6.1. Uniqueness follows from the fact that $\mathcal{A}$ is strongly dense in $B(\Gamma_a(\mathcal{H}))$. The endomorphism $\alpha$ is described concretely as follows. Let $A \in B(\Gamma_s(\mathcal{H}))$ be given. Consider the operator $A \otimes 1$ on $\Gamma_a(\mathcal{H}) \otimes \Gamma_a(Ker(V^*))$. The isometry $V$ identifies $\mathcal{H}$ with $Ran(V)$. The operator $\alpha(A)$ is defined as the operator on $\Gamma_a(\mathcal{H})$ obtained from $A \otimes 1$ after we make the following identifications.

$$\Gamma_a(\mathcal{H}) \otimes \Gamma_a(Ker(V^*)) \simeq \Gamma_a(RanV) \otimes \Gamma_a(Ker(V^*)) \simeq \Gamma_a(RanV \oplus Ker(V^*)) \simeq \Gamma_a(\mathcal{H}).$$

Note that if we perform all the identifications, we obtain

$$a(\xi) \otimes 1 \simeq a(V\xi) \otimes 1 = a(V\xi) \otimes 1 + \Gamma(-1) \otimes a(0) \simeq a(V\xi \oplus 0) \simeq a(V\xi).$$

Thus $\alpha(a(\xi)) = a(V\xi)$ for $\xi \in \mathcal{H}$. The proof is now complete. □

**Theorem 6.7.** *Let $\mathcal{H}$ be a Hilbert space and $V := \{V_x\}_{x \in P}$ be an isometric representation of $P$ on $\mathcal{H}$. Then there exists a unique $E_0$-semigroup $\alpha^V := \{\alpha_x\}_{x \in P}$ on $B(\Gamma_a(\mathcal{H}))$ such that*

$$\alpha_x(a(\xi)) = a(V_x\xi)$$

*for $\xi \in \mathcal{H}$. The $E_0$-semigroup $\alpha^V$ is called the CAR flow associated to the isometric representation $V$.*

As the proof is similar to the proof of Theorem 6.3, we omit the proof. The linear span of Weyl operators is replaced by the algebra $\mathcal{A}$. We need the following exercise.

**Exercise 6.3.** *Prove that maps $\mathcal{H} \ni \xi \to a(\xi) \in B(\Gamma_a(\mathcal{H}))$ and $\mathcal{H} \ni \xi \to a(\xi)^* \in B(\Gamma_a(\mathcal{H}))$ are continuous where $\mathcal{H}$ is given the norm topology and $B(\Gamma_a(\mathcal{H}))$ is given the strong operator topology.*

Suppose $P = [0, \infty)$. It was proved by Robinson and Powers in [24] that the CAR flow associated to the module $[0, \infty)$ of multiplicity $k$ is conjugate to the CCR flow of index $k$. Let us write the CCR flow associated to an isometric representation $V$ by $\alpha^V_{ccr}$ and the CAR flow associated to an isometric representation $V$ by $\alpha^V_{car}$. It is recently proved by R. Srinivasan in [32] that when $P$ is a higher dimensional cone, $\alpha^V_{ccr}$ and $\alpha^V_{car}$ need not be cocycle conjugate.

## 7. Measurability issues

In this section, we state the results due to Murugan which are fundamental to the subject. For, the results ensure that CCR and CAR flows are genuine $E_0$-semigroups, i.e. they are continuous in the appropriate sense (See Theorem 6.3 and 6.7). Recall the $P$ is a closed subsemigroup of a second countable locally compact topological group $G$ containing the identity $e$. We have assumed the interior of $P$, denoted $\Omega$, is dense in $P$. Murugan's result is the following.

**Theorem 7.1** (Murugan)**.** *Let $\alpha := \{\alpha_x\}_{x \in P}$ be a family of unital normal endomorphisms of $B(\mathcal{H})$. Assume that*

(1) *for $x, y \in P$, $\alpha_x \circ \alpha_y = \alpha_{xy}$, $\alpha_e = Id$, and*
(2) *for $T \in \mathcal{L}^1(\mathcal{H})$ and $A \in B(\mathcal{H})$, the map $P \ni x \to Tr(\alpha_x(A)T) \in \mathbb{C}$ is measurable.*

*Then $\alpha$ is an $E_0$-semigroup.*

In short, there is no distinction between measurable $E_0$-semigroups and continuous $E_0$-semigroups. Using the fact that finite rank operators are dense in $\mathcal{L}^1(\mathcal{H})$, we see immediately that (2) is equivalent to the following condition.

$(2)'$ For $\xi, \eta \in \mathcal{H}$ and $A \in B(\mathcal{H})$, the map $P \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$ is measurable.

Let us indicate Murugan's proof when the semigroup $P = G$.

**Lemma 7.2.** *Let $\alpha$ be a normal endomorphism of $B(\mathcal{H})$. Then there exists a unique contraction $\beta : \mathcal{L}^1(\mathcal{H}) \to \mathcal{L}^1(\mathcal{H})$ such that for $T \in \mathcal{L}^1(\mathcal{H})$ and $A \in B(\mathcal{H})$,*

$$Tr(\beta(T)A) = Tr(T\alpha(A)).$$

*Proof.* Fix $T \in \mathcal{L}^1(\mathcal{H})$. The normality of $\alpha$ implies that the map

$$B(\mathcal{H}) \ni A \to Tr(T\alpha(A)) \in \mathbb{C}$$

is $\sigma$-weakly continuous. Hence there exists a unique trace class operator, which we denote by $\beta(T)$, such that $Tr(\beta(T)A) = Tr(T\alpha(A))$. It is routine to check that $\beta$ is the desired map. □

Let us recall a few facts regarding **vector valued integration**. Suppose $E$ is a separable Banach space and let $(X, \mathcal{B})$ be a measurable space.

(1) A map $f : X \to E$ is said to be *weakly measurable* if $\phi \circ f$ is measurable for every $\phi \in E^*$.
(2) Suppose $f : X \to E$ is weakly measurable. Then the map $X \ni x \to ||f(x)|| \in \mathbb{C}$ is measurable. This is because since $E$ is separable, the unit ball of $E^*$ w.r.t. to the weak $*$-topology is a compact metrizable space.
(3) Let $\mu$ be a measure on $(X, \mathcal{B})$ and $f : X \to E$ be a weakly measurable map. We say that $f$ is integrable w.r.t $\mu$ if $x \to ||f(x)||$ is integrable. Suppose $f$ is integrable. Define $F : E^* \to \mathbb{C}$ by

$$F(\phi) = \int \phi(f(x))d\mu(x).$$

An application of the Krein-Smulian theorem implies that $F$ is weak $*$-continuous. Thus there exists a unique element, denoted $\int f(x)d\mu(x) \in E$, such that

$$\phi(\int f(x)d\mu(x)) = \int \phi(f(x))d\mu(x).$$

We call $\int f(x)d\mu(x)$, the integral of $f$ w.r.t the measure $\mu$. The $\int$ satisfies the usual linearilty properties and DCT.

Assume $P = G$ and let $\mu$ be a left Haar measure on $G$. Let $\alpha := \{\alpha_x\}_{x \in G}$ be a family of unital normal endomorphisms of $B(\mathcal{H})$ satisfying (1) and (2). For $x \in P$, let $\beta_x$ be the contraction corresponding to $\alpha_x$ given by the previous lemma. Observe that $\beta_x\beta_y = \beta_{yx}$. Note that $\mathcal{L}^1(\mathcal{H})$ is a separable Banach space. For $f \in C_c(G)$ and $T \in \mathcal{L}^1(\mathcal{H})$, define

$$T(f) = \int f(x)\beta_x(T)d\mu(x).$$

Note that for $T \in \mathcal{L}^1(\mathcal{H})$, the map $G \ni x \to \beta_x(T) \in \mathcal{L}^1(\mathcal{H})$ is weakly measurable. Thus the integral $T(f)$ makes sense.

**Exercise 7.1.** *Use Hahn-Banach theorem and the separability of $\mathcal{L}^1(\mathcal{H})$ to show that the set $\{T(f) : f \in C_c(G), T \in \mathcal{L}^1(\mathcal{H})\}$ is total in $\mathcal{L}^1(\mathcal{H})$.*

**Exercise 7.2.** *Let $f \in C_c(G)$ and $T \in \mathcal{L}^1(\mathcal{H})$ be given. Use the dominated convergence theorem to show that if $x_n \to x$, $\beta_{x_n}(T(f)) \to \beta_x(T(f))$ weakly. In other words, given $A$ in $B(\mathcal{H})$, the map $P \ni x \to Tr(\beta_x(T(f))A) \in \mathbb{C}$ is continuous.*

Combining the above two exercises and the fact the $\{\beta_x\}_{x\in G}$ is uniformly bounded, we obtain for $T \in \mathcal{L}^1(\mathcal{H})$ and $A \in B(\mathcal{H})$, the map

$$P \ni x \to Tr(\beta_x(T)A) = Tr(T\alpha_x(A)) \in \mathbb{C}$$

is continuous. Hence $\alpha$ is an $E_0$-semigroup. The proof of the general case is similar. One replaces $C_c(G)$ by $C_c(\Omega)$. For more details, we refer the reader to [20].

Here is an application of Theorem 7.1.

**Proposition 7.3.** *Let $\alpha := \{\alpha_x\}_{x\in P}$ and $\beta := \{\beta_x\}_{x\in P}$ be $E_0$-semigroups on $B(\mathcal{H})$ and $B(\mathcal{K})$ respectively. Then $\{\alpha_x \otimes \beta_x\}_{x\in P}$ is an $E_0$-semigroup on $B(\mathcal{H}\otimes\mathcal{K})$ and we denote it by $\alpha\otimes\beta$.*

*Proof.* The only thing that requires verification is the continuity property. One uses the Kaplansky density theorem argument employed in the proof of Theorem 6.3. The linear span of $\{A \otimes B : A \in B(\mathcal{H}), B \in B(\mathcal{K})\}$ takes the role of the linear span of Weyl operators here. □

**Exercise 7.3.** *Let $V$ and $W$ be isometric representations of $P$. Then $V \oplus W$ is an isometric representation of $P$. Show that the "CCR functor" takes direct sum to tensor product, i.e. $\alpha^{V\oplus W}$ is conjugate to $\alpha^V \otimes \alpha^W$.*

We need one more result of Murugan in the sequel. Just like there is no difference between measurable and continuous $E_0$-semigroups, there is no distinction between measurable and continuous cocycles.

**Proposition 7.4.** *Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup. Suppose $U := \{U_x\}_{x\in P}$ is family of unitaries satisfying the following conditions.*

(1) *For $x, y \in P$, $U_x\alpha_x(U_y) = U_{xy}$, and*
(2) *for $\xi, \eta \in \mathcal{H}$, the map $P \ni x \to \langle U_x\xi|\eta\rangle \in \mathbb{C}$ is measurable.*

*Then $U$ is strongly continuous, i.e. $U$ is an $\alpha$-cocycle.*

## 8. Measure theoretic preliminaries

We collect here the necessary preliminaries concerning measurable field of Hilbert spaces ([12]) and on standard Borel spaces ([5]). For proofs, the reader is referred to [12] and [5]. Let $(X, \mathcal{B})$ be a measurable space and $\{\mathcal{H}_x\}_{x\in X}$ be a family of separable Hilbert spaces. We assume $\mathcal{H}_x \neq 0$ for every $x \in X$. Consider the disjoint union $\coprod_{x\in X} \mathcal{H}_x$. A map

$f : X \to \coprod_{x\in X} \mathcal{H}_x$ such that $f(x) \in \mathcal{H}_x$ is called a section. Note that the set of sections is a vector space.

**Definition 8.1.** *Let $(X, \mathcal{B})$ be a measurable space. Suppose $\{\mathcal{H}_x\}_{x\in X}$ is a family of Hilbert spaces and $\Gamma$ is a subset of the set of sections. We say that $\big(\{\mathcal{H}_x\}_{x\in X}, \Gamma\big)$ is a measurable field of Hilbert spaces if*

1. *given $f, g \in \Gamma$, the map $X \ni x \to \langle f(x)|g(x)\rangle \in \mathbb{C}$ is measurable,*
2. *if $g$ is a section and $\langle g|f\rangle$ is measurable for every $f \in \Gamma$ then $g \in \Gamma$, and*
3. *there exists a countable set $\{f_n\}_{n\in\mathbb{N}}$ of $\Gamma$ such that the set $\{f_n(x) : n \in \mathbb{N}\}$ is total in $\mathcal{H}_x$ for every $x \in X$.*

*The space $\Gamma$ is called the space of measurable sections of the field $\coprod_{x\in X} \mathcal{H}_x$.*

Let $\coprod_{x\in X} \mathcal{H}_x$ be a measurable field of Hilbert spaces. Observe that (1) and (2) implies that the space of measurable sections forms a module over the algebra of complex valued measurable functions on $X$. Here is an example of a measurable field that we need later.

Let $\mathcal{H}$ be a separable Hilbert space. A map $f : X \to \mathcal{H}$ is said to be weakly measurable if for every $\xi \in \mathcal{H}$, the map $X \ni x \to \langle f(x)|\xi\rangle \in \mathbb{C}$ is measurable. Let $\{p(x)\}_{x\in X}$ be a family of projections which is weakly measurable, i.e. for $\xi, \eta \in \mathcal{H}$, the map $X \ni x \to \langle p(x)\xi|\eta\rangle \in \mathbb{C}$ is measurable. Set $\mathcal{H}_x = Ran(p(x))$. Define $\Gamma$ to be the set of all weakly measurable maps $f : X \to \mathcal{H}$ such that $f(x) \in \mathcal{H}_x$.

**Proposition 8.2.** *With the foregoing notation, the pair $\big(\{\mathcal{H}_x\}_{x\in X}, \Gamma\big)$ is a measurable field of Hilbert spaces.*

*Proof.* Let $\xi_1, \xi_2, \cdots$ be an orthonormal basis for $\mathcal{H}$. Define $f_n \in \Gamma$ by $f_n(x) = p(x)e_n$. With this, we leave the proof to the reader. $\square$

We need the following theorem in the sequel. For a proof, the reader is referred to Prop. 7.2.7 of [12]. Let $\big(\{\mathcal{H}_x\}_{x\in X}, \Gamma\big)$ be a measurable field of Hilbert spaces. Let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Define $d : X \to \mathbb{N}_\infty$ by $d(x) = \dim \mathcal{H}_x$.

**Proposition 8.3.** *With the foregoing notation, we have the following.*

1. *The dimension function $d$ is measurable.*
2. *There exists a sequence $\{u_k\} \in \Gamma$ such that for $x \in X$, $\{u_k(x)\}_{k=1}^{d(x)}$ is an orthonormal basis for $\mathcal{H}_x$ and $u_k(x) = 0$ if $k > d(x)$.*

**Exercise 8.1.** *Let $\{u_k\}$ be as in the previous proposition. Show that for a section $f$, $f \in \Gamma$ if and only $x \to \langle f(x)|u_k(x)\rangle$ is measurable for every $k$.*

Next we review the basics of Standard Borel spaces. The best reference for this is Chapter 3 of Arveson's remarkable little book "Invitation to $C^*$-algebras".

**Definition 8.4.** *A topological space is said to be Polish if it is homeomorphic to a complete separable metric space.*

(1) Compact metric spaces are Polish.
(2) Any closed subset of a Polish space is Polish.
(3) Any open subset of a Polish space is Polish.

A pleasant consequence of (1) and (3) is that a second countable locally compact Hausdorff space is Polish. The measurable structure that we consider on a topological space is always the Borel $\sigma$-algebra, i.e. the $\sigma$-algebra generated by its open subsets.

**Definition 8.5.** *Let $(X, \mathcal{B})$ be a measurable space. We say that $X$ is standard if there exists a Polish space $Y$ and a Borel subset $E$ of $Y$ such that $X$ is isomorphic to $E$ where the measurable structure on $E$ is the one induced by the Borel $\sigma$-algebra of $Y$, i.e. the measurable subsets of $E$ are the Borel sets of $Y$ contained in $E$.*

Measurable subsets of a standard Borel space are usually called *Borel subsets* and measurable maps are called *Borel*. A nice feature of standard Borel spaces is the following theorem which is Theorem 3.3.2 of [5].

**Proposition 8.6.** *Let $X$ and $Y$ be standard Borel spaces and $f : X \to Y$ be measurable.*

(1) *If $f$ is* 1-1, *then $f$ maps Borel sets to Borel sets.*
(2) *If $f$ is a bijection then $f^{-1}$ is measurable.*

Next we discuss the measurable structure that we impose on the algebra of bounded operators on a Hilbert space. Let $\mathcal{H}$ be a separable Hilbert space. Endow $B(\mathcal{H})$ with the weak operator topology and endow $B(\mathcal{H})$ with the Borel $\sigma$-algebra associated to the weak operator topology.

**Lemma 8.7.** *The map $B(\mathcal{H}) \ni T \to ||T|| \in [0, \infty)$ is measurable.*

*Proof.* Let $D$ be a countable dense subset of the unit ball of $\mathcal{H}$. Then

$$||T|| = \sup_{\xi, \eta \in D} |\langle T\xi | \eta \rangle|$$

for $T \in B(\mathcal{H})$. Since for $\xi, \eta \in \mathcal{H}$, the map $B(\mathcal{H}) \ni T \to \langle T\xi|\eta\rangle \in \mathbb{C}$ is continuous, it follows from the above equality that the map $B(\mathcal{H}) \ni T \to ||T|| \in \mathbb{C}$ is measurable. The proof is now complete. $\square$

**Lemma 8.8.** *Let $(X, \mathcal{B})$ be a measurable space and $f : X \to B(\mathcal{H})$ be a map. The following are equivalent.*

(1) *The map $f$ is measurable.*
(2) *For $\xi, \eta \in \mathcal{H}$, the map $X \ni x \to \langle f(x)\xi|\eta\rangle \in \mathbb{C}$ is measurable.*

*Proof.* By the definition of the weak operator topology, it follows that for $\xi, \eta \in \mathcal{H}$, the map $B(\mathcal{H}) \ni T \to \langle T\xi|\eta\rangle \in \mathbb{C}$ is continuous. Hence (1) implies (2). Now assume (2). We claim that the map $X \ni x \to ||f(x)|| \in \mathbb{C}$ is measurable. Let $\{\xi_n\}_{n=1}^{\infty}$ be a countable dense subset of the unit ball of $\mathcal{H}$. Note that for $x \in X$,

$$||f(x)|| = \sup_{m,n} |\langle f(x)\xi_m|\xi_n\rangle|.$$

Hence $x \to ||f(x)||$ is measurable. This proves our claim.

Let $B_r$ be the closed ball of $B(\mathcal{H})$, centered at 0 with radius $r$. Then $f^{-1}(B_r)$ is measurable for each $r > 0$. Let $U$ be a weakly open subset of $B(\mathcal{H})$. The hypothesis implies that if $U$ is a basic open set, then $f^{-1}(U)$ is measurable. Observe that $f^{-1}(U) = \bigcup_{n=1}^{\infty} f^{-1}(U \cap B_n)$. It suffices to prove that $f^{-1}(U \cap B_n)$ is measurable for every $n$. Write $U = \bigcup_{\alpha} U_\alpha$ with $U_\alpha$ being basic open sets. Note that $B_n$ is a compact, second countable space when given the weak operator topology. Hence there exists a countable collection $\{U_m\}$ of $\{U_\alpha\}$ such that $U \cap B_n = \bigcup_{m=1}^{\infty} U_m \cap B_n$. Now $f^{-1}(U \cap B_n) = \bigcap_{m=1}^{\infty} f^{-1}(U_m) \cap f^{-1}(B_n)$. Hence $f^{-1}(U \cap B_n)$ is measurable. This completes the proof. □

I learnt the following proof from Murugan.

**Proposition 8.9.** *The space $B(\mathcal{H})$ is a standard Borel space.*

*Proof.* Let $B$ be the closed unit ball of $B(\mathcal{H})$ and endow $B$ with the weak opeartor topology. Then $B$ is metrisable and compact. Thus $B$ is a Polish space. Denote the open unit ball by $B^0$. Note that $B^0 = \bigcup_{n=1}^{\infty}\{T \in B : ||T|| \leq 1 - \frac{1}{n}\}$. Thus $B^0$ is a Borel subset of $B$. Define $f : B(\mathcal{H}) \to B^0$ by

$$f(T) = \frac{T}{1 + ||T||}.$$

It is clear that $f$ is a bijection. Lemma 8.7 and Lemma 8.8 imply that $f$ is measurable. Note that $f^{-1} : B^0 \to B(\mathcal{H})$ is given by $f^{-1}(S) = \frac{S}{1-||S||}$. Thus $f^{-1}$ is measurable. Hence $B(\mathcal{H})$ is a standard Borel space. This completes the proof. □

**Remark 8.10.** *One could have started with any one of the four topologies on $B(\mathcal{H})$ i,e. weak, strong, $\sigma$-weak or the norm topology. Although the topologies are different, the measurable structure induced by each one of the topologies is the same.*

With these measure theoretic preliminaries, we can proceed towards the study of product systems.

## 9. Product system of an $E_0$-semigroup

First we need a lemma which identifies the intertwining space of the composition of two normal endomorphisms. For a subset $S \subset B(\mathcal{H})$, we write $[S]$ for the closed linear span of $S$ where the closure is taken in the norm topology. Let $\mathcal{H}$ be a separable Hilbert space.

**Lemma 9.1.** *Let $\alpha$ and $\beta$ be unital, normal endomorphisms of $B(\mathcal{H})$. Denote the intertwining space of $\alpha$ and $\beta$ by $E$ and $F$ respectively. Then the intertwining space of $\alpha \circ \beta$ is $[EF]$ where $EF = \{TS : T \in E, S \in F\}$. Moreover the map*

$$E \otimes F \ni T \otimes S \to TS \in [EF]$$

*is a unitary between the Hilbert spaces $E \otimes F$ and $[EF]$.*

*Proof.* It is clear that $[EF]$ is contained in the intertwining space of $\alpha \circ \beta$. Suppose $R$ is an element in the intertwining space of $\alpha \circ \beta$ such that $R \perp EF$. Let $\{W_j\}$ be an orthonormal basis for $E$ and $\{V_i\}$ be an orthonormal basis for $F$.

Note that for every $i, j$, $R^*W_jV_i = 0$. Post multiply by $V_i^*W_j^*$ and sum over $i$ to obtain $R^*W_j\beta(1)W_j^* = 0$. Now sum over $j$ to obtain $R^*\alpha(\beta(1)) = 0$. But $R$ is an element in the intertwining space of $\alpha \circ \beta$. Thus $R^*\alpha(\beta(1)) = R^*$. Hence $R = 0$. This proves that $[EF]$ is the intertwining space of $\alpha \circ \beta$. The second assertion is immediate. □

Let us recall Exercise 3.1. Let $\alpha$ and $\beta$ be unital normal endomorphisms of $B(\mathcal{H})$. Denote the intertwining space of $\alpha$ and $\beta$ by $E$ and $F$ respectively. Denote the space of intertwiners between $\alpha$ and $\beta$ by $L(\alpha, \beta)$, i.e.

$$L(\alpha, \beta) = \{T \in B(\mathcal{H}) : T\alpha(A) = \beta(A)T \text{ for } A \in B(\mathcal{H})\}.$$

Given $C \in L(\alpha, \beta)$, define $\theta_C \in B(E, F)$ by $\theta_C(T) = CT$. Then $C \to \theta_C$ identifies $L(\alpha, \beta)$ with $B(E, F)$. How do we recover $C$ from $\theta_C$ ? The map $E \otimes \mathcal{H} \ni T \otimes \xi \to T\xi \in \mathcal{H}$, denoted $U_E$, and the map $F \otimes \mathcal{H} \ni S \otimes \eta \to S\eta \in \mathcal{H}$, denoted $U_F$, are unitaries. Then $C = U_F(\theta_C \otimes 1)U_E^*$. Note that $C$ is a unitary if and only $\theta_C$ is a unitary. We will repeatedly use the identification $C \to \theta_C$ in what follows.

**Definition 9.2.** *By a product system over $P$, we mean a standard Borel space $E$, together with a measurable surjection $p : E \to P$ such that the following holds.*

1. *For $x \in P$, $E(x) := p^{-1}(x)$ is a non-zero separable Hilbert space. For the identity element $e$, $E(e) = \mathbb{C}$.*
2. *There exists an associative multiplication $E \times E \ni (u, v) \to uv \in E$ such that $p(uv) = p(u)p(v)$. Moreover the multiplication $E \times E \ni (u, v) \to uv$ is measurable.*
3. *The multiplication maps $E(e) \times E(x) \ni (\lambda, u) \to \lambda u$ and $E(x) \times E(e) \ni (u, \lambda) \to u.\lambda$ are just scalar multiplication on $E(x)$.*

(4) *For* $x,y \in P$, *there exists a unitary* $u_{x,y} : E(x) \otimes E(y) \to E(xy)$ *that satisfy* $u_{x,y}(u \otimes v) = uv$.
(5) *Let* $\Delta = \{(u,v) \in E \times E : p(u) = p(v)\}$. *The maps* $\Delta \ni (u,v) \to u+v \in E$ *and* $\Delta \ni (u,v) \to \langle u|v\rangle \in \mathbb{C}$ *are measurable.*
(6) *The scalar multiplication* $\mathbb{C} \times E \ni (\lambda,u) \to \lambda u \in E$ *is measurable.*
(7) *Let* $\Gamma = \{s : P \to E : s$ *is measurable and* $s(x) \in E(x)$ *for* $x \in P\}$. *Then* $(E,\Gamma)$ *is a measurable field of Hilbert spaces.*

We simply write a product system as $E = \coprod_{x\in P} E(x)$.

**Definition 9.3.** *Let* $E := \coprod_{x\in P} E(x)$ *and* $F := \coprod_{x\in P} F(x)$ *be product systems over* $P$. *We say that* $E$ *and* $F$ *are isomorphic if for* $x \in P$, *there exists a unitary operator* $\theta_x : E(x) \to F(x)$ *such that*

(1) *the map* $\theta := \coprod_{x\in P} \theta_x$ *is measurable,* $\theta^{-1}$ *is measurable, and .*
(2) *the map* $\theta$ *preserves the product rule, i.e. for* $x,y \in P$, $u \in E(x)$, $v \in E(y)$,
$$\theta_{xy}(uv) = \theta_x(u)\theta_y(v).$$

**Remark 9.4.** *Keep the notation of the previous definition. Since* $E$ *and* $F$ *are standard Borel spaces, it suffices to require that* $\theta$ *is measurable.*

Our goal in this section is to associate a product system to an $E_0$-semigroup and to show that the associated product system determines the $E_0$-semigroup, up to cocycle conjugacy. Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$. For $x \in P$, let $E(x)$ be the intertwining space of $\alpha_x$. Let $E := \coprod_{x\in P} E(x)$, i.e.
$$E = \{(x,T) : x \in P, T \in E(x)\}.$$

**Lemma 9.5.** *The set* $E$ *is a Borel subset of* $P \times B(\mathcal{H})$, *where* $P \times B(\mathcal{H})$ *is given the product structure.*

*Proof.* It suffices to show that $\{(x,T) : x \in P, T \in E(x), ||T|| \leq r\}$ is measurable for every $r \in \mathbb{N}$. Fix a natural number $r$. Let $B$ the closed unit ball of $B(\mathcal{H})$ of radius $r$ centered at 0. We claim that $E_r := \{(x,T) : x \in P, T \in E(x), ||T|| \leq r\}$ is closed in $P \times B$. Note that $P \times B$ is metrisable when $B$ is given the weak operator topology. Let $(x_n,T_n)$ be a sequence in $E_r$ such that $(x_n,T_n) \to (x,T)$. Let $A \in B(\mathcal{H})$ be given.

By Lemma 4.2, it follows that $\alpha_{x_n}(A^*)$ converges strongly to $\alpha_x(A^*)$. Now $T_n^* \to T^*$ in WOT implies that $T_n^*\alpha_{x_n}(A^*) \to T^*\alpha_x(A^*)$ in WOT. On the other hand, $A^*T_n^* = T_n^*(\alpha_{x_n}(A^*))$ since $(x_n,T_n) \in E$ and $A^*T_n^* \to A^*T^*$ in WOT. Thus $T^*\alpha_x(A^*) = A^*T^*$.

Taking adjoints, we obtain $TA = \alpha_x(A)T$. Thus $(x,T) \in E_r$. This proves our claim. Hence the proof. $\square$

Keep the foregoing notation.

(1) Recall that for $x \in P$, $E(x)$ is a separable Hilbert space where the inner product on $E(x)$ is given by $\langle T|S\rangle = S^*T$.
(2) Lemma 9.1 implies that for $x, y \in P$, $S \in E(x)$, $T \in E(y)$, we have $ST \in E(xy)$ and the map $E(x) \otimes E(y) \ni S \otimes T \to ST \in E(xy)$ is a unitary.
(3) It is clear that $E(e) = \mathbb{C}$.

Define a multiplication on $E$ as follows:

$$(x,S)(y,T) = (xy, ST).$$

It is clear that the above multiplication is associative. Let $p : E \to P$ be the projection onto the first coordinate. Then the pair $(E,p)$ satisfies the first six axioms of Definition 9.2. We make repeated use of Lemma 8.8 to verify the various measurability conditions.

**Theorem 9.6.** *The set $E$, together with the structures described above, is a product system.*

The only non-trivial thing to prove is (7) of Definition 9.2. We will prove after a series of Lemmas. Let $d : P \to [0,\infty]$ be defined by $d(x) = \dim E(x)$.

**Lemma 9.7.** *The function $d$ is measurable.*

*Proof.* Observe the following. Suppose $\alpha$ is a normal endomorphism having the representation $\alpha(A) = \sum_{i=1}^{d} V_i A V_i^*$ as in Theorem 2.3. Note that $d$ is the dimension of the intertwining space of $\alpha$. Let $Q$ be a rank one projection. Then $\alpha(Q)$ is a sum of $d$ orthogonal rank one projections. Thus $d = \dim Ran(\alpha(Q))$. But for a projection $Q$, $\dim(Ran(Q)) = Tr(Q)$.

Fix an orthonormal basis $\{\xi_1, \xi_2, \cdots\}$ for $\mathcal{H}$ and a rank one projection $Q$. Then, for $x \in P$, $d(x) = Tr(\alpha_x(Q)) = \displaystyle\sum_{i=1}^{\infty} \langle \alpha_x(Q)\xi_i|\xi_i\rangle$. This proves that the function $d$ is measurable. $\square$

Partition the semigroup $P$ as follows. For $k \in \mathbb{N}_\infty$, let $P_k = \{x \in P : d(x) = k\}$. Then $P_k$ is a measurable subset of $P$ and $P := \coprod_{k=1}^{\infty} P_k$. Fix $k \in \mathbb{N}_\infty$ and a point $x_0 \in P_k$. Note that for $x \in P_k$, $E(x)$ and $E(x_0)$ are of the same dimension. Let $\theta_x : E(x_0) \to E(x)$ be a unitary. Then there exists a unitary $U_x \in L(\alpha_{x_0}, \alpha_x) \subset B(\mathcal{H})$ such that $\theta_x = \theta_{U_x}$ (See the paragraph following Lemma 9.1). This means that $\alpha_x(A) = U_x \alpha_{x_0}(A) U_x^*$. The point we wish to stress is that we can choose the family $\{U_x\}_{x \in P_k}$ in a measurable way. This is the content of the next proposition.

**Proposition 9.8.** *Let $k \in \mathbb{N}_\infty$ and $x_0 \in P_k$ be given. Then there exists a family of unitaries $\{U_x\}_{x \in P_k}$ such that*

1. *for $\xi, \eta \in \mathcal{H}$, the map $P_k \ni x \to \langle U_x \xi | \eta \rangle \in \mathbb{C}$ is measurable, and*
2. *for $x \in P_k$ and $A \in B(\mathcal{H})$, $\alpha_x(A) = U_x \alpha_{x_0}(A) U_x^*$.*

**Lemma 9.9.** *Let $(X, \mathcal{B})$ be a measurable space and $\{p(x)\}_{x \in X}$ be a weakly measurable family of projections of constant rank. Fix $x_0 \in X$. Then there exists a weakly measurable family of partial isometries $\{w(x)\}_{x \in X}$ such that for every $x \in X$, $w(x)^* w(x) = p(x)$ and $w(x) w(x)^* = p(x_0)$.*

*Proof.* Consider the measurable field of Hilbert space $(\{\mathcal{H}_x\}_{x \in X}, \Gamma)$ of Prop. 8.2. Let $d$ be the rank of $p(x_0)$. Apply Prop. 8.3 to find a sequence $\{u_k\}_{k=1}^d \in \Gamma$ such that for every $x \in X$, $\{u_k(x)\}_{k=1}^d$ is an orthonormal basis for $Ran(p(x))$. For $x \in X$, let $w(x)$ be the unique partial isometry such that $w(x)^* w(x) = p(x)$, $w(x) w(x)^* = p(x_0)$ and $w(x)(u_k(x)) = u_k(x_0)$.

To show that $\{w(x)\}_{x \in X}$ is weakly measurable, it suffices to show that, for every $k, \ell$, the map $x \to \langle w(x)^* u_k(x_0) | u_\ell(x_0) \rangle$ is measurable. But the latter is just $\langle u_k(x) | u_\ell(x_0) \rangle$ which is measurable. This completes the proof. □

*Proof of Prop. 9.8.* Let $Q := \theta_{\xi_0, \xi_0}$ be a rank one projection. Choose a weakly measurable family of partial isometries $\{w_x\}_{x \in P_k}$ such that $w_x w_x^* = \alpha_x(Q)$ and $w_x^* w_x = \alpha_{x_0}(Q)$. We claim the following.

1. For every $x \in P_k$, $\{\alpha_x(A) w_x \xi : A \in B(\mathcal{H}), \xi \in \mathcal{H}\}$ is total in $\mathcal{H}$, and
2. for $A, B \in B(\mathcal{H})$, $\xi, \eta \in \mathcal{H}$ and $x \in P_k$,

$$\langle \alpha_x(A) w_x \xi | \alpha_x(B) w_x \eta \rangle = \langle \alpha_{x_0}(A) w_{x_0} \xi | \alpha_{x_0}(B) w_{x_0} \eta \rangle = \langle A \xi_0 | B \xi_0 \rangle \langle \alpha_{x_0}(Q) \xi | \eta \rangle.$$

Note that $\{AQB : A, B \in B(\mathcal{H})\}$ is total in $B(\mathcal{H})$ with respect to the $\sigma$-weak topology. Thus $\{\alpha_x(A) \alpha_x(Q) \alpha_x(B) \xi : A, B \in B(\mathcal{H}), \xi \in \mathcal{H}\}$ is total in $\mathcal{H}$. But the set $\{\alpha_x(A) w_x w_x^* \alpha_x(B) \xi : A, B \in B(\mathcal{H}), \xi \in \mathcal{H}\}$ is contained in $\{\alpha_x(A) w_x \xi : A \in B(\mathcal{H}), \xi \in \mathcal{H}\}$. As a consequence, we have the totatlity of $\{\alpha_x(A) w_x \xi : A \in B(\mathcal{H}), \xi \in \mathcal{H}\}$ in $\mathcal{H}$. This proves (1).

Observe that $QTQ = \langle T\xi_0|\xi_0\rangle Q$ for every $T \in B(\mathcal{H})$. Let $x \in P_k$, $\xi, \eta \in \mathcal{H}$ and $A, B \in B(\mathcal{H})$ be given. Calculate as follows to observe that

$$\begin{aligned}
\langle \alpha_x(A)w_x\xi|\alpha_x(B)w_x\eta\rangle &= \langle w_x^*\alpha_x(B^*A)w_x\xi|\eta\rangle \\
&= \langle w_x^*w_xw_x^*\alpha_x(B^*A)w_xw_x^*w_x\xi|\eta\rangle \\
&= \langle w_x^*\alpha_x(QB^*AQ)w_x\xi|\eta\rangle \\
&= \langle A\xi_0|B\xi_0\rangle\langle w_x^*\alpha_x(Q)w_x\xi|\eta\rangle \\
&= \langle A\xi_0|B\xi_0\rangle\langle w_x^*w_xw_x^*w_x\xi|\eta\rangle \\
&= \langle A\xi_0|B\xi_0\rangle\langle \alpha_{x_0}(Q)\xi|\eta\rangle
\end{aligned}$$

This proves (2).

For $x \in P_k$, let $U_x$ be the unique unitary such that $U_x(\alpha_{x_0}(A)w_{x_0}\xi) = \alpha_x(A)w_x\xi$ for $A \in B(\mathcal{H})$ and $\xi \in \mathcal{H}$. Let $A \in B(\mathcal{H})$ and $x \in P_k$ be given. Calculate as follows to observe that for $B \in B(\mathcal{H})$, $\xi \in \mathcal{H}$,

$$\begin{aligned}
U_x\alpha_{x_0}(A)(\alpha_{x_0}(B)w_{x_0}\xi) &= U_x\alpha_{x_0}(AB)w_{x_0}\xi \\
&= \alpha_x(AB)w_x\xi \\
&= \alpha_x(A)\alpha_x(B)w_x\xi \\
&= \alpha_x(A)U_x(\alpha_{x_0}(B)w_{x_0}\xi).
\end{aligned}$$

The totality of the set $\{\alpha_{x_0}(B)w_{x_0}\xi\}$ in $\mathcal{H}$ implies that $U_x\alpha_{x_0}(A) = \alpha_x(A)U_x$. The measurability of $\{U_x\}_{x\in P_k}$ follows from the measurability of $\{w_x\}_{x\in P_k}$. This completes the proof. □

*Proof of Theorem 9.6.* For every $k \in \mathbb{N}_\infty$, pick a point $x_k \in P_k$. Fix an orthonormal basis, say $\{V_{ki}\}_{i=1}^k$ of $E(x_k)$. Let $\{U_x^{(k)}\}$ be a family of unitaries as in Proposition 9.8. Define for $i = 1, 2, \cdots$, $W_i(x) \in E(x)$ as follows:

$$W_i(x) = U_x^{(k)}V_{ki} \text{ if } x \in P_k \text{ and } i \leq k. \tag{9.4}$$

Set $V_i(x) = W_i(x)$ if $i \leq d(x)$ or else 0. Note that for for every $i$, $V_i$ is weakly measurable. Moreover for $x \in P$, $\{V_i(x)\}_{i=1}^{d(x)}$ is an orthonormal basis for $E(x)$. It is now straightforward to check that $E$, together with its measurable sections, forms a measurable field of Hilbert spaces. □

The product system $E$ constructed in 9.6 is called *the product system associated with the $E_0$-semigroup $\alpha$*. If we wish to emphasize the dependence of $E$ on $\alpha$, we write it as $E_\alpha$. Next we show that the product system $E_\alpha$ completely determines $\alpha$.

**Theorem 9.10.** *Let $\alpha := \{\alpha_x\}_{x\in P}$ and $\beta := \{\beta_x\}_{x\in P}$ be $E_0$-semigroups on $B(\mathcal{H})$. Then the following are equivalent.*

(1) *The $E_0$-semigroups $\alpha$ and $\beta$ are cocycle conjugate.*
(2) *The product systems $E_\alpha$ and $E_\beta$ are isomorphic.*

*Proof.* Suppose $\alpha$ and $\beta$ are cocycle conjugate. Let $\{U_x\}_{x\in P}$ be an $\alpha$-cocycle such that for $x \in P$, $\beta_x = Ad(U_x) \circ \alpha_x$. For $x \in P$, define $\theta_x : E_\alpha(x) \to E_\beta(x)$ by $\theta_x(T) = U_xT$. Then $\theta_x$ is a unitary for every $x \in P$. Let $x, y \in P$ and $S \in E_\alpha(x)$, $T \in E_\alpha(y)$ be given. Calculate as follows to observe that

$$\begin{aligned}
\theta_{xy}(ST) &= U_{xy}ST \\
&= U_x\alpha_x(U_y)ST \\
&= U_xSU_yT \\
&= \theta_x(S)\theta_y(T).
\end{aligned}$$

This proves that $\theta := \coprod_{x\in P} \theta_x$ is multiplicative. The fact that $\theta$ is Borel follows from the fact that $\{U_x\}_{x\in P}$ is a weakly measurable family of unitaries.

Now assume that $E_\alpha$ and $E_\beta$ are isomorphic. Let $\theta := \coprod_{x\in P} \theta_x : E_\alpha \to E_\beta$ be an isomorphism. Since $\theta_x : E_\alpha(x) \to E_\beta(x)$ is a unitary, it follows that there exists a unique unitary $U_x$ such that $Ad(U_x) \circ \alpha_x = \beta_x$ and $\theta_x(T) = U_xT$.

We claim that $\{U_x\}_{x\in P}$ is an $\alpha$-cocycle. For $S \in E_\alpha(x)$, $T \in E_\alpha(y)$, the equality

$$U_{xy}(ST) = \theta_{xy}(ST) = \theta_x(S)\theta_y(T) = U_xSU_yT = U_x\alpha_x(U_y)ST$$

and the fact that $\{ST\xi : S \in E_\alpha(x), T \in E_\alpha(x), \xi \in \mathcal{H}\}$ is total in $\mathcal{H}$ implies that for $x, y \in P$, $U_{xy} = U_x\alpha_x(U_y)$ i.e. $\{U_x\}_{x\in P}$ satisfies the cocycle equation.

Let $d$ be the dimension function of $E_\alpha$. Choose a sequence of measurable sections $\{V_k\}$ of $E_\alpha$ such that

(1) $V_k(x) = 0$ if $k > d(x)$
(2) for $x \in P$, $\{V_k(x)\}_{k=1}^{d(x)}$ is an orthonormal basis for $E_\alpha(x)$.

Fix $x \in P$. Observe that $U_xV_k(x) = \theta_x(V_k(x))$. Post multiply by $V_k(x)^*$ and add up to $d(x)$ to see that for $x \in P$,

$$U_x = \sum_{k=1}^{d(x)} \theta_x(V_k(x))V_k(x)^*.$$

The map $\theta$ is measurable. This implies that $x \to \theta_x(V_k(x))$ is measurable for every $k$. A consequence of the above equality is that $x \to U_x$ is weakly measurable. Appealing to Proposition 7.4, we conclude that $\{U_x\}$ is an $\alpha$-cocycle. By definition, $\beta_x = Ad(U_x) \circ \alpha_x$. Hence $\beta$ is a cocycle perturbation of $\alpha$. This completes the proof. $\square$

Let us specialise to the case of $E_0$-semigroups which are made of "automorphisms". Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup such that for every $x \in P$, $\alpha_x$ is an automorphism.

Let $E_\alpha$ be the product system associated to $\alpha$. Since $\alpha_x$ is an automorphism, it follows that $\dim E_\alpha(x) = 1$ for every $x \in P$. This is because if $Q$ is a rank one projection then $\alpha_x(Q)$ is a minimal and hence a rank one projection.

Note that $E_\alpha$ is a measurable field of Hilbert spaces, each of whose fibres are one dimensional. Thus, there exists a weakly measurable family of unitaries $\{U_x\}_{x\in P}$ such that $\alpha_x = Ad(U_x)$. Choose such a family. Note that $\alpha_x \circ \alpha_y = \alpha_{xy}$ implies that for $x, y \in P$, there exists $\omega(x, y) \in \mathbb{T}$ such that

$$U_{xy} = \omega(x, y)U_xU_y.$$

It is routine to verify that $\omega$ is a Borel multiplier. Summarising our discussion so far, we have the following.

**Theorem 9.11.** *Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$ such that $\alpha_x$ is an automorphism for every $x \in P$. Then there exists a Borel multiplier $\omega$ and a $\omega$-projective unitary representation $\{U_x\}_{x\in P}$ on $\mathcal{H}$ such that for $x \in P$,*

$$\alpha_x(A) = U_xAU_x^*.$$

If we specialise the above theorem to the case when $P = [0, \infty)$, we obtain Wigner's theorem. First we need the following fact. The proof can be found in [29].

**Lemma 9.12.** *Let $\omega$ be a Borel multiplier on $\mathbb{R}$. Then there exists a Borel function $f : \mathbb{R} \to \mathbb{T}$ such that $\omega(s, t) = f(s)f(t)f(s + t)^{-1}$ for $s, t \in \mathbb{R}$.*

**Theorem 9.13** (Wigner's theorem)**.** *Let $\alpha := \{\alpha_t\}_{t\geq 0}$ be an $E_0$-semigroup on $B(\mathcal{H})$. Suppose that $\alpha_t$ is an automorphism for every $t \geq 0$. Then there exists a strongly continuous unitary representation $\{U_t\}_{t\geq 0}$ such that $\alpha_t = Ad(U_t)$.*

*Proof.* First we extend the $E_0$-semigroup $\alpha$ to an $E_0$-semigroup over $\mathbb{R}$. Let $t \in \mathbb{R}$ be given. Write $t = r - s$ with $r, s \geq 0$. Set $\alpha_t = \alpha_r \circ \alpha_s^{-1}$. It is routine to check that $\alpha_t$ depends only on $t$ and $\{\alpha_t\}_{t\in\mathbb{R}}$ is an $E_0$-semigroup over $\mathbb{R}$ which extends $\alpha$.

By Theorem 9.11, it follows that there exists a Borel multiplier $\omega$ on $\mathbb{R}$ and a $\omega$-projective unitary representation $\{W_t\}_{t\in\mathbb{R}}$ such that $\alpha_t = Ad(W_t)$. Choose a Borel map $f : \mathbb{R} \to \mathbb{T}$ such that $\omega(s, t) = \frac{f(s)f(t)}{f(s+t)}$. Set $U_t = f(t)W_t$. Then $\{U_t\}_{t\in\mathbb{R}}$ is a weakly measurable family of unitaries such that $U_sU_t = U_{s+t}$. Imitating the proof of 7.1, we can conclude that $\{U_t\}_{t\in\mathbb{R}}$ is strongly continuous. But $W_t$ and $U_t$ differ by a scalar of modulus 1. Thus for $t \in \mathbb{R}$, $\alpha_t = Ad(W_t) = Ad(U_t)$. This completes the proof. □

**Arveson's question:** One of the most natural and also the fundamental question in Arveson's programme of $E_0$-semigroups is the following.

*Is every product system, over $P$, isomorphic to a product system associated with an $E_0$-semigroup over $P$ ? In other words, is the study of $E_0$-semigroups and the study of product systems the same ?*

This was answered in the affirmative when $P = [0, \infty)$ by Arveson. Arveson's first proof makes essential use of the $C^*$-algebra, called *the spectral $C^*$-algebra of $E$*, associated to the product system $E$ and a deep analysis of its state space. Later Skeide in [30] found a much simpler proof. Arveson himself found a simpler proof in [8].

Imitating Arveson's proof in [8], the author and Murugan, have shown in [21] and [19], that the above question has an affirmative answer when

(1) $P$ is isomorphic to a finitely generated subsemigroup of $\mathbb{Z}^d$, and
(2) when $P$ is a closed convex cone in $\mathbb{R}^d$ which is pointed and spanning.

The proof of (1) and (2) are essentially the same. But at the time of writing, it was not clear how to bring both proofs under a common framework. Now it is possible to unify the proofs and this was carried out in the first draft of Murugan's thesis.

For what kind of semigroups, the map

$$\alpha \to E_\alpha$$

is a bijection between the class of $E_0$-semigroups and the class of product systems ? Is the statement true for the semigroup of natural numbers with mutliplication ? Note that the semigroup of natural numbers with multiplication is not finitely generated as there are infinitely many primes.

It is relatively simple to prove that a product system with a unit, over an Ore semigroup, is isomorphic to a product system of an $E_0$-semigroup. The construction is based on an inductive limit procedure and is due to Arveson. To explain this construction, we need to talk about representations of product systems.

**Definition 9.14.** *Let $E$ be a product system over $P$. Suppose $\mathcal{H}$ is a separable Hilbert space and $\phi : E \to B(\mathcal{H})$ is a map. We say that $\phi$ is a representation of $E$ on $\mathcal{H}$ if*

(1) *the map $\phi$ is measurable,*
(2) *for $x \in P$, $u, v \in E(x)$, $\phi(v)^*\phi(u) = \langle u|v\rangle$, and*
(3) *for $x, y \in P$, $u \in E(x)$ and $v \in E(y)$, $\phi(uv) = \phi(u)\phi(v)$.*

*A representation $\phi$ is called essential if $[\phi(E(x))\mathcal{H}] = \mathcal{H}$ for every $x \in P$.*

Let $E$ be a product system over $P$.

**Lemma 9.15.** *Suppose $\phi : E \to B(\mathcal{H})$ is a representation. Then $\phi$ restricted to each fibre is linear.*

*Proof.* Fix $x \in P$. Let $u, v \in E(x)$ be given. Set $T = \phi(u+v) - \phi(u) - \phi(v)$. For $w \in E(x)$, $\xi, \eta \in \mathcal{H}$, calculate as follows to observe that

$$\begin{aligned}
\langle T\xi|\phi(w)\eta\rangle &= \langle \phi(u+v)\xi|\phi(w)\eta\rangle - \langle \phi(u)\xi|\phi(w)\eta\rangle - \langle \phi(v)\xi|\phi(w)\eta\rangle \\
&= \langle \phi(w)^*\phi(u+v)\xi|\eta\rangle - \langle \phi(w)^*\phi(u)\xi|\eta\rangle - \langle \phi(w)^*\phi(v)\xi|\eta\rangle \\
&= \langle u+v|w\rangle\langle \xi|\eta\rangle - \langle u|w\rangle\langle \xi|\eta\rangle - \langle v|w\rangle\langle \xi|\eta\rangle \\
&= 0.
\end{aligned}$$

Thus $T\xi$ is orthogonal to every vector of the form $\phi(w)\eta$, $w \in E(x)$, $\eta \in \mathcal{H}$. But $T\xi$ is a linear combination of such vectors. Thus $\langle T\xi|T\xi\rangle = 0$ for every $\xi \in \mathcal{H}$. This proves that $T = 0$. This implies that $\phi$ is additive. In a similar fashion, the fact that $\phi$ preserves scalar multiplication can be proved. $\Box$

Let $E$ be a product system over $P$ and $\phi : E \to B(\mathcal{H})$ be a representation. For $x \in P$, note that Condition (2) of Defn. 9.14 and Lemma 9.15 imply that $\phi(E(x))$ is an intertwining space. For $x \in P$, let $\alpha_x$ be the unique normal endomorphism of $B(\mathcal{H})$ whose intertwining space is $\phi(E(x))$. Condition (3) of Defn. 9.14 implies that $[\phi(E(x))\phi(E(y))] = \phi(E(xy))$. Lemma 9.1 implies that $\alpha_x \circ \alpha_y = \alpha_{xy}$. Thus $\alpha^\phi := \{\alpha_x\}_{x \in P}$ is a semigroup of normal endomorphisms. Note that $\alpha_x$ is unital if and only if $[\phi(E(x))\mathcal{H}] = \mathcal{H}$. Thus $\alpha^\phi$ is a semigroup of unital normal endomorphisms if and only if $\phi$ is essential.

Assume that $\phi$ is essential. Let $d$ be the dimension function of $E$ and choose a sequence $\{e_k\}$ of measurable sections such that

(1) for $x \in P$, $\{e_k(x)\}_{k=1}^{d(x)}$ is an orthonormal basis for $E(x)$, and
(2) $e_k(x) = 0$ if $k > d(x)$.

Note that for $x \in P$ and $A \in B(\mathcal{H})$,

$$\alpha_x(A) = \sum_{k=1}^{d(x)} \phi(e_k(x))A\phi(e_k(x))^*.$$

The above equation together with the measurability of $\phi$ imply that for $A \in B(\mathcal{H})$, $\xi, \eta \in \mathcal{H}$, the map $P \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$ is measurable. Now Theorem 7.1 implies that $\alpha^\phi$ is an $E_0$-semigroup. By definition, $\phi$ is an isomorphism between $E$ and the product system associated to $\alpha^\phi$.

Thus the question of Arveson is equivalent to the following question.

**Question:** Let $E$ be a product system over $P$. Does $E$ admit an essential representation on a separable Hilbert space ?

We pose another interesting question that is worth investigating. It is easy to see that every product system always has a representation "the left regular representation".

Suppose $E$ is a product system. Let $\mathcal{H} := L^2(P, E)$ where the measure that we take on $P$ is a left Haar measure on $G$. For $x \in P$ and $u \in E(x)$, define a bounded operator $\phi(u)$ on $L^2(E)$ by the following formula:

$$\phi(u)(f)(y) := \begin{cases} uf(x^{-1}y) & \text{if } x^{-1}y \in P, \\ \\ 0 & \text{if } x^{-1}y \notin P. \end{cases} \tag{9.5}$$

It is routine to verify that $\phi$ is a representation of $E$.

For $f \in L^1(E)$, define

$$\phi(f) := \int \phi(f(x))dx.$$

**Definition 9.16.** *The spectral $C^*$-algebra of $E$, denoted $C^*(E)$, is defined as the $C^*$-algebra generated by $\{\phi(f) : f \in L^1(E)\}$.*

Let us understand the spectral $C^*$-algebra when $E$ is the trivial product system, i.e. each fibre is $\mathbb{C}$ and the multiplication rule is the usual multiplication. Assume further more that $G$ and $P$ are discrete. In this case, we can identify $L^2(E)$ with $\ell^2(P)$. Let $V := \{V_a\}_{a \in P}$ be the "left" regular representation of $P$ on $\ell^2(P)$. Then $C^*(E)$ is the $C^*$-algebra generated by $\{V_a\}_{a \in P}$ and it is denoted by $C^*_{red}(P)$. The $C^*$-algebra $C^*_{red}(P)$ is called the *reduced $C^*$-algebra* of the semigroup $P$. The study of semigroup $C^*$-algebras and the computation of the associated $K$-groups have received much attention in the recent years. We refer the reader to [10] for a more comprehensive account of the theory of semigroup $C^*$-algebras.

In the continuous case, when the semigroup $P$ is Ore, i.e. $P^{-1}P = G = PP^{-1}$, the spectral $C^*$-algebra associated to the trivial product system is called the *Wiener-Hopf algebra* and is denoted $\mathcal{W}(P)$. The systematic study of the Wiener-Hopf algebra from the groupoid perspective was initiated in [18] and further developed by Nica in [22] and Hilgert and Neeb in [13].

Taking mileage out of the groupoid approach to Wiener-Hopf algebras, the following statements were established.

(1) Let $P$ be a polyhedral cone in $\mathbb{R}^d$. Then the $K$-groups of $\mathcal{W}(P)$ vanish. This was proved by Alldridge in [2].
(2) Let $P$ be a symmetric cone in a Euclidean space. Then the $K$-groups of $\mathcal{W}(P)$ vanish. This was proved by the author in [33] by appealing to the theory of Jordan algebras. A prototypical example of a symmetric cone is the cone of positive matrices in the space of symmetric matrices. For more on symmetric cones and Jordan algebras, the reader is referred to [11].

**Question:** Suppose $P$ is either a polyhedral or a symmetric cone and $E$ is a product system over $P$. Is $K_i(C^*(E)) = 0$ for $i = 0, 1$?

The answer is yes in the one dimensional case. This was due to Zacharias([38]) for type $II$ product sytems and Hirshberg([14]) for general product systems.

**Remark 9.17.** *The vanishing results of the $K$-groups of the Wiener-Hopf algebras in the higher dimensional case makes essential use of the groupoid realisation of the Wiener-Hopf algebras. In view of this, it is of intrinsic interest to know wheter $C^*(E)$ has a groupoid realisation or not when the fibres are infinite dimensional.*

## 10. Arveson's Inductive limit construction

We show, in this section, that if the product system contains a *unit*, then it always has an essential representation on a separable Hilbert space for nice semigroups. This construction is due to Arveson and is based on an inductive limit procedure. Inductive limit constructions work well when the semigroup $P$ is Ore.

First let us define the notion of units. Product systems are bundles of Hilbert spaces. The classical Serre-Swan theorem asserts that knowing a vector bundle over a compact space is equivalent to knowing its sections. Keeping this mind, it is natural to look at sections of a product system. But a product system is more than just a bundle and it has a product structure. Thus, one is naturally led to look for sections which are multiplicative. This leads us to the notion of a unit.

**Definition 10.1.** *Let $E$ be a product system over $P$. By a unit of $E$, we mean a family $\{u_x\}_{x \in P}$ such that*

1. *for $x \in P$, $u_x \in E(x)$ and $u_x \neq 0$,*
2. *for $x, y \in P$, $u_{xy} = u_x u_y$, and*
3. *the map $P \ni x \to u_x \in E$ is measurable.*

A unit is thus a nowhere vanishing measurable section which is multiplicative. We must mention here that there are uncountably many examples of one parameter product systems which do not have any unit. A product system is said to be *spatial* if it possesses a unit.

Let us now recall the notion of a right Ore semigroup. The semigroup $P$ is said to be **right Ore** in $G$ if $PP^{-1} = G$. Suppose that $P$ is right Ore in $G$. Note that $\Omega\Omega^{-1} = G$. For $x, y \in G$, we write $x \leq y$ if there exists $a \in P$ such that $y = xa$. Similarly, we write $x < y$ if $y = xa$ for some $a \in \Omega$. Note that $\leq$ is a pre-order on $G$.

Examples listed in the beginning of Chapter 3 are right Ore. An example of a semigroup which is not Ore is the free semigroup on $n$ generators. Let $\mathbb{F}_n$ be the free group on $n$ generators and let $\{a_1, a_2, \cdots, a_n\}$ be the generators. Words in $\{a_1, a_2, \cdots, a_n\}$ form a semigroup which we denote by $\mathbb{F}_n^+$. Then $\mathbb{F}_n^+$ is not Ore in $\mathbb{F}_n$.

**Lemma 10.2.** *Let $P$ be right Ore in $G$. Given $x, y \in G$, there exists $z \in G$ such that $z \geq x, y$.*

*Proof.* Let $x, y \in G$ be given. It suffices to show that $xP \cap yP \neq \emptyset$. Choose $a, b \in P$ such that $x^{-1}y = ab^{-1}$. Then $xa = yb$. Thus $xP \cap yP \neq \emptyset$. This completes the proof. □

For the rest of this section, we assume that $PP^{-1} = G$. Before we describe Arveson's inductive limit construction, let us review the construction of the inductive limit of Hilbert spaces.

Let $\Lambda$ be a directed set. Suppose $(\{\mathcal{H}_\alpha\}_{\alpha \in \Lambda}, \{V_{\beta\alpha}\})$ is a *directed system of Hilbert spaces*, i.e.

(1) for every $\alpha \in \Lambda$, $\mathcal{H}_\alpha$ is a Hilbert space
(2) for $\alpha \leq \beta$, the map $V_{\beta\alpha} : \mathcal{H}_\alpha \to \mathcal{H}_\beta$ is an isometry, and
(3) the isometries $\{V_{\beta\alpha}\}$ satisfy the following compatibility relation: for $\alpha \leq \beta \leq \gamma$,

$$V_{\gamma\beta}V_{\beta\alpha} = V_{\gamma\alpha}.$$

Suppose we are given a directed system of Hilbert spaces as above. We claim that there exists a Hilbert space $\mathcal{H}_\infty$ and isometries $i_\alpha : \mathcal{H}_\alpha \to \mathcal{H}_\infty$ with the following properties.

(a) For $\alpha \leq \beta$, $i_\beta V_{\beta\alpha} = i_\alpha$, and
(b) The union $\bigcup_\alpha i_\alpha \mathcal{H}_\alpha$ is dense in $\mathcal{H}_\infty$.

Consider the set $H := \{(\xi, \alpha) : \xi \in \mathcal{H}_\alpha, \alpha \in \Lambda\}$. Define an equivalence relation on $H$ as follows: Let $(\xi, \alpha), (\eta, \beta) \in H$ be given. We say that $(\xi, \alpha) \sim (\eta, \beta)$ if there exists $\gamma \geq \alpha, \beta$ such that $V_{\gamma\alpha}\xi = V_{\gamma\beta}\eta$. Verify that $\sim$ is an equivalence relation on $H$. Then $H$ has an inner product structure where the addition, scalar multiplication and inner product are given as follows:

$$[(\xi, \alpha)] + [(\eta, \beta)] = [(V_{\gamma\alpha}\xi + V_{\gamma\beta}\eta, \gamma)]$$
$$\lambda[(\xi, \alpha)] = [(\lambda\xi, \alpha)]$$
$$\langle [(\xi, \alpha)] | [(\eta, \beta)] \rangle = \langle V_{\gamma\alpha}\xi | V_{\gamma\beta}\eta \rangle$$

where $\gamma$ is any element such that $\gamma \geq \alpha, \beta$. We let $\mathcal{H}_\infty$ be the completion of $H$. Let $i_\alpha : \mathcal{H}_\alpha \to \mathcal{H}_\infty$ be defined by $i_\alpha(\xi) = [(\xi, \alpha)]$. It is clear from the definition that $i_\alpha$ is an isometry and $\bigcup_\alpha i_\alpha \mathcal{H}_\alpha$ is dense in $\mathcal{H}_\infty$. It follows from the defintion that $V_{\beta\alpha} i_\alpha = i_\beta$ if $\beta \geq \alpha$.

**Exercise 10.1.** *Suppose there exists another Hilbert space $\widetilde{\mathcal{H}}_\infty$ and isometries $j_\alpha : \mathcal{H}_\alpha \to \widetilde{\mathcal{H}}_\infty$ such that $j_\beta V_{\beta\alpha} = j_\alpha$ and $\bigcup_\alpha j_\alpha \mathcal{H}_\alpha$ is dense in $\widetilde{\mathcal{H}}_\infty$. Show that there exists a unique unitary $U : \mathcal{H}_\infty \to \widetilde{\mathcal{H}}_\infty$ such that $U \circ i_\alpha = j_\alpha$.*

In view of the above Exercise, we can talk about "the inductive limit" of the system $(\{\mathcal{H}_\alpha\}_{\alpha\in\Lambda}, \{V_{\beta\alpha}\})$.

**Exercise 10.2.** *Let $\mathcal{H}_\infty$ be the inductive limit of the Hilbert spaces $\mathcal{H}_\alpha$ with connecting isometries $\{V_{\beta\alpha}\}$. Suppose $\mathcal{K}$ is another Hilbert space and for each $\alpha \in \Lambda$, there exists a bounded linear map $T_\alpha : \mathcal{H}_\alpha \to \mathcal{K}$ such that $T_\beta V_{\beta\alpha} = T_\alpha$ and $\sup_\alpha ||T_\alpha|| < \infty$. Show that there exists a unique bounded linear map $T : \mathcal{H}_\infty \to \mathcal{K}$ such that $T \circ i_\alpha = T_\alpha$.*

For our purposes, it is important to know under what conditions the inductive limit of separable Hilbert spaces is separable. Let $\Lambda$ be a directed set and $J$ be a subset of $\Lambda$. We say that $J$ is *cofinal* in $\Lambda$ if given $\alpha \in \Lambda$ there exists $\beta \in J$ such that $\beta \geq \alpha$. Let $(\{\mathcal{H}_\alpha\}_{\alpha\in\Lambda}, \{V_{\beta\alpha}\})$ be a directed system of Hilbert spaces. Denote the inductive limit by $\mathcal{H}_\infty$.

**Lemma 10.3.** *Keep the foregoing notation. Assume that $\mathcal{H}_\alpha$ is separable for each $\alpha \in \Lambda$. Suppose there exists a countable subset $J$ of $\Lambda$ such that $J$ is cofinal in $\Lambda$. Then $\mathcal{H}_\infty$ is separable.*

*Proof.* For $\beta \in J$, let $D_\beta$ be a countable dense subset of $\mathcal{H}_\beta$. Observe that if $\alpha \leq \beta$ then $i_\alpha\mathcal{H}_\alpha \subset i_\beta\mathcal{H}_\beta$. Note that the cofinality of $J$ implies that $\bigcup_{\alpha\in\Lambda} i_\alpha\mathcal{H}_\alpha = \bigcup_{\beta\in J} i_\beta\mathcal{H}_\beta$. But $\bigcup_{\beta\in J} i_\beta\mathcal{H}_\beta$ is dense in $\mathcal{H}_\infty$ and $D_\beta$ is dense in $\mathcal{H}_\beta$. Consequently, the countable set $\bigcup_{\beta\in J} i_\beta(D_\beta)$ is dense in $\mathcal{H}_\infty$. This completes the proof. □

For the rest of this section, assume that $P$ is right Ore, i.e. $PP^{-1} = G$. Observe that $\Omega$ is an ideal in $P$, i.e. $\Omega P \subset \Omega$ and $P\Omega \subset \Omega$. Also note that $\Omega\Omega^{-1} = G$.

**Lemma 10.4.** *The directed set $(G, \leq)$ admits a countable cofinal set $D$ such that $D \subset \Omega$.*

*Proof.* Observe that $G = \Omega\Omega^{-1}$. This implies that $\{a\Omega^{-1} : a \in \Omega\}$ is an open cover of $G$. But $G$ is second countable. Thus the cover $\{a\Omega^{-1} : a \in \Omega\}$ has a countable subcover. Let $a_1, a_2, \cdots$ be a sequence in $\Omega$ such that $\bigcup_{n=1}^{\infty} a_n\Omega^{-1} = G$. Set $D = \{a_n : n \in \mathbb{N}\}$. Then $D$ is cofinal in $G$. This completes the proof. □

**Lemma 10.5.** *There exists Borel maps $s, t : G \to \Omega$ such that $g = s(g)t(g)^{-1}$ for every $g \in G$.*

*Proof.* Let $a_n$ be a sequence in $\Omega$ such that $\bigcup_n a_n\Omega^{-1} = G$. Let $A_n = a_n\Omega^{-1}$ and $B_n$ be the "disjointification" of $A_n$, i.e. define $B_n$ inductively as follows: $B_1 = A_1$, $B_n = A_n \backslash \cup_{i=1}^{n-1} B_i$. Then $B_n$ forms a disjoint family of measurable subsets of $G$ and $\coprod_{n=1}^{\infty} B_n = G$. Define the map $s : G \to \Omega$ such that $s(g) = a_n$ if $g \in B_n$. Clearly $s$ is measurable. Define $t : G \to G$ by $t(g) = g^{-1}s(g)$. Clearly if $g \in B_n$, $g^{-1}s(g) \in \Omega$. Thus the range of $t$ is contained in $\Omega$. It is clear that $t$ is measurable and $g = s(g)t(g)^{-1}$ for every $g \in G$. This completes the proof. □

**Proposition 10.6.** *Let $E$ be a spatial product system over $P$. Then there exists a unit $\{u_x\}_{x \in P}$ of $E$ such that $\langle u_x|u_x\rangle = 1$ for every $x \in P$.*

*Proof.* Let $v := \{v_x\}_{x \in P}$ be a unit of $E$. Define $\chi(x) = \langle v_x|v_x\rangle$ for $x \in P$. Since $v$ is nowhere vanishing, it follows that $\chi$ takes values in the multiplicative group $(0, \infty)$. Let $x, y \in P$ be given. Calculate as follows to observe that

$$\begin{aligned}
\chi(xy) &= \langle v_{xy}|v_{xy}\rangle \\
&= \langle v_x v_y|v_x v_y\rangle \\
&= \langle v_x|v_x\rangle\langle v_y|v_y\rangle \\
&= \chi(x)\chi(y).
\end{aligned}$$

Thus $\chi$ is a homomorphism. For $x \in P$, set $u_x := (\chi(x))^{-\frac{1}{2}} v_x$. Then $u := \{u_x\}_{x \in P}$ is a unit with the desired property. □

Let $E$ be a spatial product system and $u := \{u_x\}_{x \in P}$ be a unit such that $\langle u_x|u_x\rangle = 1$ for every $x \in P$. Consider the directed set $(P, \leq)$. For $x \in P$, consider the Hilbert space $E(x)$ and for $x \leq y$, let $V_{y,x} : E(x) \to E(y)$ be the isometry defined by $V_{y,x}(e) = eu_{x^{-1}y}$. Then $(\{E(x)\}_{x \in P}, \{V_{y,x}\})$ is a directed system of Hilbert spaces. Denote the inductive limit by $\mathcal{H}$. For $x \in P$, let $i_x$ be the inclusion of $E(x)$ into $\mathcal{H}$. By Lemma 10.3 and by Lemma 10.4, it follows that $\mathcal{H}$ is separable.

Let $x \in P$ and $e \in E(x)$ be given. Let $\phi(e)$ be the unique bounded linear operator on $\mathcal{H}$ such that $\phi(e)i_y(f) = i_{xy}(ef)$ for $y \in P$ and $f \in E(y)$. The existence of such a bounded operator on $\mathcal{H}$ is guaranteed by Exercise 10.2. It is clear that for $x \in P$, $e, \widetilde{e} \in E(x)$, $\phi(\widetilde{e})^*\phi(e) = \langle e|\widetilde{e}\rangle$. Also for $x, y \in P$ and $e \in E(x)$, $f \in E(y)$, $\phi(ef) = \phi(e)\phi(f)$.

We claim that for $x \in P$, $\phi(E(x))\mathcal{H}$ is total in $\mathcal{H}$. Fix $x \in P$. Let $y \geq x$ be given. Write $y = xa$ with $a \in P$. Suppose $f \in E(y)$. It suffices to prove that $i_y(f) \in \overline{\phi(E(x))\mathcal{H}}$. Since $E(x)E(a)$ is dense in $E(y)$, it suffices to verify when $f$ is of the form $f = eu$ where $e \in E(x)$ and $u \in E(a)$. Then clearly $i_y(f) = \phi(e)i_a(u)$. Thus $i_y(E(y))$ is contained in $\phi(E(x))\mathcal{H}$. But $\{i_y(E(y)) : y \geq x\}$ is total in $\mathcal{H}$. As a consequence, it follows that $\phi(E(x))\mathcal{H}$ is total in $\mathcal{H}$. This proves our claim.

Thus the only non trivial thing that requires proof is the measurability of $\phi$. First, we need a lemma.

**Lemma 10.7.** *Let $E$ be a product system and let $\phi : E \to B(\mathcal{H})$ be a map satisfying* (2) *and* (3) *of Definition 9.14. Then the following are equivalent.*

(1) *The map $\phi$ is measurable.*
(2) *For every Borel section $s$ of $E$, $\phi \circ s$ is weakly measurable.*

*Proof.* It is clear that (1) implies (2). Assume that (2) holds. Let $d$ be the dimension function of $E$. Suppose $\{u_k\}_{k=1}^{\infty}$ is a sequence of Borel sections such that if $k > d(x)$, $u_k(x) = 0$ and $\{u_k(x)\}_{k=1}^{d(x)}$ forms an orthonormal basis for $E(x)$. Denote the projection of $E$ onto $P$ by $p$.

Note that for every $k$, the map $E \ni u \to \langle u|u_k(p(u))\rangle \in \mathbb{C}$ is measurable. Observe that for $u \in E$,

$$u = \sum_{k=1}^{d(p(u))} \langle u|u_k(p(u))\rangle u_k(p(u)).$$

Since $\phi$ restricted to each fibre is a bounded linear map, it follows that for $u \in E$,

$$\phi(u) = \sum_{k=1}^{d(p(u))} \langle u|u_k(p(u))\rangle \phi(u_k(p(u))).$$

The hypothesis imply that $\phi(u_k(p(u)))$ is measurable for every $k$. The above expression together with the fact that $u \to \langle u|u_k(p(u))\rangle$ is measurable for each $k$ imply that $\phi$ is measurable. This completes the proof. $\square$

Let us now return to the discussion preceding Lemma 10.7. Keep the notation used in the discussion preceding Lemma 10.7. We claim that the map $\phi$ is measurable. Let $e : P \to E$ be a measurable section. It suffices to show that for $y, z \in P$ and $f \in E(y)$, $g \in E(z)$, the map $P \ni x \to \langle \phi(e(x))i_y(f)|i_z(g)\rangle \in \mathbb{C}$ is measurable. Let $s$ and $t$ be the maps constructed in Lemma 10.5.

Calculate as follows to observe that for $x \in P$,

$$\begin{aligned}
\langle \phi(e(x))i_y(f)|i_z(g)\rangle &= \langle i_{xy}(e(x)f)|i_z(g)\rangle \\
&= \langle i_{xyt(z^{-1}xy)}(e(x)fu_{t(z^{-1}xy)}|i_{zs(z^{-1}xy)}(gu_{s(z^{-1}xy)})\rangle \\
&= \langle e(x)fu_{t(z^{-1}xy)}|gu_{s(z^{-1}xy)}\rangle \text{ (since } xyt(z^{-1}xy) = zs(z^{-1}xy)).
\end{aligned}$$

The above expression clearly indicates that $x \to \langle \phi(e(x))i_y(f)|i_z(g)\rangle$ is measurable. This is because multiplication and taking inner products are measurable operations. Thus $\phi$ is a representation. We can summarise the above as follows.

**Theorem 10.8.** *Suppose $E$ is a spatial product system over a right Ore semigroup. Then $E$ admits an essential representation.*

## 11. Pure $E_0$-semigroups

For reasons, that are not clear to the author, a class of $E_0$-semigroups that are considered important are called *pure $E_0$-semigroups*. In the last two sections, we discuss pure $E_0$-semigroups and standard forms of $E_0$-semigroups. In particular, we discuss Alevras' work on standard forms of spatial $E_0$-semigroups. For the rest of this notes, we assume that $P$ is right Ore, i.e. $PP^{-1} = G$. Recall that we say for $x, y \in G$, $x \leq y$ ($x < y$) if $x^{-1}y \in P$ ($x^{-1}y \in \Omega$). The fact that $P$ is right Ore implies that $(G, \leq)$ and $(P, \leq)$ are directed sets.

**Definition 11.1.** *Let $\alpha := \{\alpha_x\}_{x \in P}$ be an $E_0$-semigroup. We say that $\alpha$ is pure if*

$$\bigcap_{x \in P} \alpha_x(B(\mathcal{H})) = \mathbb{C}.$$

We proceed towards deriving an equivalent definition in terms of normal states of $B(\mathcal{H})$.

**Lemma 11.2.** *Let $\alpha : B(\mathcal{H}) \to B(\mathcal{H})$ be a unital normal endomorphism. Then the image $\alpha(B(\mathcal{H}))$ is a von Neumann subalgebra of $B(\mathcal{H})$.*

*Proof.* Write $\alpha$ as $\alpha(A) = \sum_{i=1}^{d} V_i A V_i^*$ where $\{V_i\}_{i=1}^{d}$ is a family of isometries with orthogonal range projections. Let $(B_j)$ be a net in $\alpha(B(\mathcal{H}))$ such that $B_j \to B$ in the $\sigma$-weak topology. Write $B_j = \alpha(A_j)$. Note that $V_1^* B_j V_1 = V_1^* \alpha(A_j) V_1 = A_j$ and $V_1^* B_j V_1 \to V_1^* B V_1$. As a consequence, it follows that $A_j$ converges and let $A$ be the limit. Since $\alpha$ is normal, it follows that $\alpha(A_j) \to \alpha(A)$. Thus $B = \alpha(A)$. This shows that $\alpha(B(\mathcal{H}))$ is $\sigma$-weakly closed. Hence the proof. □

Let $\Lambda$ be a directed set. For $\alpha \in \Lambda$, let $M_\alpha$ be a unital von Neumann subalgebra of $B(\mathcal{H})$. Assume that $M_\alpha$ is decreasing, i.e. $M_\alpha \subset M_\beta$ if $\alpha \geq \beta$. Denote the intersection $\bigcap_{\alpha \in \Lambda} M_\alpha$ by $M_\infty$.

**Lemma 11.3.** *Let $\omega$ be a normal functional on $B(\mathcal{H})$. Then*

$$||\omega|_{M_\infty}|| = \inf_{\alpha \in \Lambda} ||\omega|_{M_\alpha}||.$$

*Proof.* Since $M_\infty \subset M_\alpha$ for every $\alpha$, it follows that $||\omega|_{M_\infty}|| \leq ||\omega|_{M_\alpha}||$ for every $\alpha$. Thus, it follows that $||\omega|_{M_\infty}|| \leq \inf\limits_{\alpha \in \Lambda} ||\omega|_{M_\alpha}||$. Suppose the inequality is strict. Choose a number $C$ in between. Then for every $\alpha \in \Lambda$, $C < ||\omega|_{M_\alpha}||$. Thus there exists $x_\alpha \in M_\alpha$ such that $||x_\alpha|| = 1$ and $|\omega(x_\alpha)| > C$. The unit ball of $B(\mathcal{H})$ is $\sigma$-weakly compact. By passing to a subnet, if necessary, we can assume that $x_\alpha$ converges to a point $x \in B(\mathcal{H})$, of norm at most 1, in the $\sigma$-weak topology. However $\{M_\alpha\}$ is decreasing and $M_\alpha$ is $\sigma$-weakly closed. This implies that $x \in M_\alpha$ for each $\alpha$. Thus $x \in M_\infty$.

Taking limit in the inequality $|\omega(x_\alpha)| > C$, we obtain $||\omega|| \geq ||\omega(x)|| \geq C$ which is a contradiction to our assumption that $C > ||\omega|_{M_\infty}||$. Hence

$$||\omega|_{M_\infty}|| = \inf_{\alpha \in \Lambda} ||\omega|_{M_\alpha}||.$$

This completes the proof. □

**Lemma 11.4.** *Let $A$ and $B$ be $C^*$-algebras and let $\pi : A \to B$ be a sujrective $*$-homomorphism. Then $\pi$ maps the unit ball of $A$* ***onto*** *the unit ball of $B$.*

*Proof.* Since $\pi$ is contractive, it follows that $||\pi(a)|| \leq 1$ if $||a|| \leq 1$. Let $b \in B$ be such that $||b|| \leq 1$.

*Case 1:* Suppose that $b$ is self-adjoint. Since $\pi$ is surjective, there exists $a \in A$ such that $\pi(a) = b$. Replacing $a$ by $\frac{a+a^*}{2}$, we can assume that there exists a self-adjoint element $a \in A$ such that $\pi(a) = b$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} x & \text{if } |x| \leq 1, \\ 1 & \text{if } x > 1, \\ -1 & \text{if } x < -1. \end{cases}$$

Note that $f(a)$ is of norm at most 1 and $\pi(f(a)) = f(\pi(a)) = f(b) = b$.

*Case 2:* Let $b \in B$ be such that $||b|| \leq 1$. Amplify $A$, $B$ and $\pi$ by $2 \times 2$ matrices. The element $\begin{bmatrix} 0 & b^* \\ b & 0 \end{bmatrix}$ is self-adjoint and has norm at most 1. Thus there exists a self-adjoint $2 \times 2$ matrix over $A$, say, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ of norm at most 1 such that $\pi\Big(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\Big) = \begin{bmatrix} 0 & b^* \\ b & 0 \end{bmatrix}$. Note that $a_{21}$ has norm at most 1 and $\pi(a_{21}) = b$. This completes the proof. $\Box$

Let $\alpha := \{\alpha_x\}_{x \in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$. For $x \in P$, let $\beta_x$ be the linear map on $\mathcal{L}^1(\mathcal{H})$ such that for $T \in \mathcal{L}^1(\mathcal{H})$ and $A \in B(\mathcal{H})$,

$$Tr(\beta_x(T)A) = Tr(T\alpha_x(A)).$$

Note that $\beta_x \circ \beta_y = \beta_{yx}$ for $x, y \in P$. Recall that $(P, \leq)$ is a directed set. For $x \in P$, let $M_x = \alpha_x(B(\mathcal{H}))$. Note that $\{M_x\}_{x \in P}$ is a decreasing net of von Neumann algebras. Denote its intersection by $M_\infty$.

**Proposition 11.5.** *Keep the foregoing notation. Then the following are equivalent.*

1. *The $E_0$-semigroup $\alpha$ is pure.*
2. *For every pair of normal states $\rho_1$ and $\rho_2$ of $B(\mathcal{H})$, the net $(||\rho_1 \circ \alpha_x - \rho_2 \circ \alpha_x||)_{x \in P}$ converges to 0.*
3. *For postive trace class operators $T_1, T_2$ with trace 1, the net $(||\beta_x \circ T_1 - \beta_x \circ T_2||)_{x \in P}$ converges to 0.*

*Proof.* The equivalence between (2) and (3) is obvious. Assume that (1) holds. Let $\rho_1$ and $\rho_2$ be normal states on $B(\mathcal{H})$. Set $\lambda = \rho_1 - \rho_2$. Observe that for $x \in P$, by Lemma

11.4, we have the equality

$$||\rho_1 \circ \alpha_x - \rho_2 \circ \alpha_x|| = ||\lambda \circ \alpha_x|| = ||\lambda|_{M_x}||.$$

By Lemma 11.3, the net $(||\lambda|_{M_x}||)_{x \in P}$ decreases to $||\lambda|_{M_\infty}||$. But $M_\infty = \mathbb{C}$ and $\lambda$ vanishes on $M_\infty$. Hence $||\rho_1 \circ \alpha_x - \rho_2 \circ \alpha_x|| = ||\lambda|_{M_x}|| \to 0$. Thus we have proved (1) $\implies$ (2).

Now suppose that (2) holds. By the Hahn-Banach theorem, it suffices to show that if $T$ is a trace class operator with trace zero, then $Tr(TA) = 0$ for every $A \in M_\infty$. Let $T$ be such an operator. We can write $T = T_1 + iT_2$ with $T_1, T_2$ self-adjoint. Hence $Tr(T_1) = 0 = Tr(T_2)$. Thus, we can assume that $T$ is self-adjoint. Let $T_+$ and $T_-$ be the positive and the negative parts of $T$. Thus $Tr(T_+) = Tr(T_-)$. By rescaling, if necessary, we can assume $Tr(T_+) = 1$. Let $\rho_+$ and $\rho_-$ be the normal states on $B(\mathcal{H})$ associated to $T_+$ and $T_-$ respectively.

By Lemma 11.3, Lemma 11.4 and the hypothesis, we conclude that $||(\rho_+ - \rho_-)|_{M_\infty}|| = 0$. Thus $\rho_+ - \rho_-$ vanishes on $M_\infty$. In other words, $Tr(TA) = 0$ for every $A \in M_\infty$. Hence $M_\infty = \mathbb{C}$. Thus the implication (2) $\implies$ (1) is proved. $\square$

Keep the foregoing notation.

**Definition 11.6.** *Let $\omega$ be a normal state on $B(\mathcal{H})$. We say that $\omega$ is absorbing for the $E_0$-semigroup $\alpha$ if for every normal state $\rho$, the net $(||\rho \circ \alpha_x - \omega||)_{x \in P}$ converges to 0.*

Let $\omega$ be a normal state on $B(\mathcal{H})$ and $T$ be the trace class operator that corresponds to $\omega$. Then $\omega$ is absorbing if and only if for every positive trace class operator $S$ with $Tr(S) = 1$, we have $(\beta_x \circ S)_{x \in P} \to T$. An absorbing state is invariant. To see this, suppose $\omega$ is an absorbing state and $T$ be the trace class operator associated to it. Then $\beta_x \circ T \to T$. Fix $a \in P$. Since $\beta_a$ is continuous, it follows that $\beta_a(\beta_x(T)) = \beta_{xa}(T) \to \beta_a(T)$. But $\{xa : x \in P\}$ is cofinal in $P$. Thus $(\beta_{xa}(T))_{x \in P} \to T$. As a consequence, we obtain $\beta_a(T) = T$. In other words, $\omega \circ \alpha_a = \omega$ for every $a \in P$.

**Remark 11.7.** *Note that Prop.11.5 implies that if an $E_0$-semigroup has an absorbing state then it is pure. But, it is not true that every pure $E_0$-semigroup has an absorbing state. We refer the reader to Section 7.3 of* [7]. *If $\alpha$ is a pure $E_0$-semigroup and $\omega$ is an invariant normal state then $\omega$ is absorbing for $\alpha$.*

Next we show that CCR flows associated to a pure isometric representation is pure. Let $V : P \to B(\mathcal{H})$ be an isometric representation. We say that the representation $V$ is *pure* if $\displaystyle\bigcap_{x \in P} V_x\mathcal{H} = \{0\}$.

**Exercise 11.1.** *Let $V : P \to B(\mathcal{H})$ be an isometric representation. The following conditions are equivalent.*

(1) *The representation* $V$ *is pure.*
(2) *The net* $(V_x^*)_{x\in P} \to 0$ *in SOT.*
(3) *The net* $(E_x)_{x\in P} \to 0$ *in SOT where* $E_x = V_xV_x^*$.

Here is an example of a pure isometric representation.

**Example 11.8.** *Let* $A$ *be a* $P$*-module and let* $V$ *be the isometric representation associated to the module* $A$. *If* $A \neq G$, *then* $V$ *is pure.*

*Proof.* Suppose $g \in \bigcap_{x\in P} V_x\mathcal{H}$. Then $g \perp Ker(V_x^*) = L^2(A\backslash xA)$. We claim that $\bigcap_{x\in P} xA = \emptyset$. Suppose $z \in \bigcap_{x\in P} xA$. Then $x^{-1}z \in A$ for every $x \in P$. But $A$ is a $P$-module and $PP^{-1} = G$. Thus $G = Gz = PP^{-1}z \subset PA \subset A$ which is a contradiction. This proves our claim.

Let $\phi$ be a compactly supported continuous function on $A$ and $K$ be its support. Note that $\{K \cap xA\}_{x\in P}$ is a decreasing family of compact sets having empty intersection. Thus there exists $x \in P$ such that $K \cap xA = \emptyset$. In other words, $K \subset A\backslash xA$. Then $\phi \in L^2(A\backslash xA)$. Consequently, $\langle g|\phi\rangle = 0$. Since $C_c(A)$ is dense in $L^2(A)$, it follows that $g = 0$. This completes the proof. □

Our next proposition states that CCR flows associated to pure isometric representations are pure. For an isometric representation $V$ of $P$ on $\mathcal{H}$, let $\alpha^V$ be the CCR flow associated to $V$. Denote the vacuum vector of the symmetric Fock space $\Gamma(\mathcal{H})$ by $v$ and the vector state $B(\Gamma(\mathcal{H})) \ni A \to \langle Av|v\rangle \in \mathbb{C}$ by $\omega$.

**Proposition 11.9.** *Suppose* $V$ *is a pure isometric representation. Then the associated CCR flow* $\alpha^V$ *is pure and the vector state given by the vacuum vector is absorbing for* $\alpha^V$.

*Proof.* We simply denote $\alpha^V$ by $\alpha := \{\alpha_x\}_{x\in P}$. The CCR relations imply that for $x \in P$ and $\xi \in Ker(V_x^*)$, $\eta \in \mathcal{H}$, $W(\xi)W(V_x\eta) = W(V_x\eta)W(\xi)$. This has the consequence that $\{W(\xi) : \xi \in Ker(V_x)^*\}$ is contained in the commutant of $\alpha_x(B(\Gamma(\mathcal{H})))$. Note that the map $\mathcal{H} \ni \xi \to W(\xi)$ is continuous when $\mathcal{H}$ is given the norm topology and $B(\Gamma(\mathcal{H}))$ is given the strong operator topology. The representation $\{V_x\}_{x\in P}$ is pure is equivalent to the fact that the increasing union $\bigcup_{x\in P} Ker(V_x)^*$ is dense in $\mathcal{H}$. The discussion so far imply that $\{W(\xi) : \xi \in \mathcal{H}\}$ is contained in the strong closure of $\bigcup_{x\in P} \alpha_x(B(\Gamma(\mathcal{H})))'$. But the linear span of $\{W(\xi) : \xi \in \mathcal{H}\}$ is strongly dense in $B(\Gamma(\mathcal{H}))$. Thus it follows that $\bigcup_{x\in P} \alpha_x(B(\Gamma(\mathcal{H})))'$ is strongly dense in $B(\Gamma(\mathcal{H}))$. Taking commutant, we obtain $\bigcap_{x\in P} \alpha_x(B(\Gamma(\mathcal{H}))) = \mathbb{C}$. Thus $\alpha$ is pure.

Since $\alpha$ is pure, to show that the vector state, denote $\omega$, given by the vacuum vector $e(0)$ is absorbing for $\alpha$, it suffices to show that it is invariant. A direct calculation reveals that for $x \in P$ and $\xi \in \mathcal{H}$,

$$\langle \alpha_x(W(\xi))e(0)|e(0)\rangle = \langle W(\xi)e(0)|e(0)\rangle.$$

The invariance of $\omega$ follows from the fact that $\alpha_x$ is normal and the linear span of Weyl operators is $\sigma$-weakly dense in $B(\Gamma(\mathcal{H}))$. This completes the proof. □

## 12. Standard form

In this section, we discuss Alevras' work ([1]) on standard forms of $E_0$-semigroups. For a unit vector $\xi \in \mathcal{H}$, we denote the corresponding vector state by $\omega_\xi$, i.e. for $A \in B(\mathcal{H})$, $\omega_\xi(A) = \langle A\xi|\xi\rangle$.

**Definition 12.1.** *Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$. We say that $\alpha$ is in standard form if there exists a unit vector $\xi \in \mathcal{H}$ such that $\omega_\xi$ is absorbing for $\alpha$. Equivalently, there exists a unit vector $\xi \in \mathcal{H}$ such that $\omega_\xi$ is $\alpha$-invariant and $\alpha$ is pure.*

Let $\alpha$ be a normal unital endomorphism of $B(\mathcal{H})$ and $\xi$ be a unit vector. Denote the intertwining space of $\alpha$ by $E$. With the foregoing notation, we have the following.

**Lemma 12.2.** *The following are equivalent.*

1. *The state $\omega_\xi$ is $\alpha$-invariant, i.e. for $A \in B(\mathcal{H})$, $\omega_\xi(\alpha(A)) = \omega_\xi(A)$.*
2. *The vector $\xi$ is a common eigen vector for $\{T^* : T \in E\}$*

*Proof.* Suppose that (1) holds. Define an operator $S : \mathcal{H} \to \mathcal{H}$ by $SA\xi = \alpha(A)\xi$ for $A \in B(\mathcal{H})$. The fact that $\omega_\xi$ is $\alpha$-invariant implies that $S$ is well defined. A direct calculation reveals that $S \in E$. Note that $S\xi = \xi$. Let $T \in E$ be given. Now $T^*\xi = T^*S\xi = \langle S|T\rangle\xi$. Thus $\xi$ is an eigen vector of $T^*$. Thus the implication (1) $\implies$ (2) is proved.

Now suppose (2) holds. Write $\alpha(A) = \sum_{i=1}^{d} V_iAV_i^*$ where $\{V_i\}_{i=1}^{d}$ is a family of isometries with orthogonal range projections. Let $\lambda_i \in \mathbb{C}$ be such that $V_i^*\xi = \lambda_i\xi$. Note that

$$1 = \omega_\xi(1) = \omega_\xi(\sum_{i=1}^{d} V_iV_i^*) = \sum_{i=1}^{d}\langle V_iV_i^*\xi|\xi\rangle = \sum_{i=1}^{d}|\lambda_i|^2\langle\xi|\xi\rangle = \sum_{i=1}^{d}|\lambda_i|^2.$$

Let $A \in B(\mathcal{H})$ be given. Observe that

$$\omega_\xi(\alpha(A)) = \sum_{i=1}^{d}\langle V_iAV_i^*\xi|\xi\rangle = \sum_{i=1}^{d}\langle AV_i^*\xi|V_i^*\xi\rangle = \sum_{i=1}^{d}|\lambda_i|^2\langle A\xi|\xi\rangle = \langle A\xi|\xi\rangle.$$

Thus $\omega_\xi$ is $\alpha$ invariant. This completes the proof.

Keep the foregoing notation. Let $Q$ be the orthogonal range projection onto the one dimensional space spanned by $\xi$.

**Lemma 12.3.** *The projection onto the closed subspace $E\xi$ is $\alpha(Q)$.*

*Proof.* Write $\alpha(A) = \sum_{i=1}^{d} V_iAV_i^*$. It is clear that for $\eta \in \mathcal{H}$, $\alpha(Q)\eta = \sum_{i=1}^{d} V_iQV_i^*\eta \in E\xi$. Let $T \in E$ be given. Noting the fact that $\{V_i\}_{i=1}^{d}$ is an orthonormal basis for $E$, Observe that

$$\alpha(Q)T\xi = \sum_{i=1}^{d} V_iQV_i^*T\xi = \sum_{i=1}^{d}\langle T|V_i\rangle V_iQ\xi = (\sum_{i=1}^{d}\langle T|V_i\rangle V_i)\xi = T\xi.$$

This completes the proof. □

Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$ and $\xi$ be a unit vector. Denote the projection onto the 1-dimensional subspace spanned by $\xi$ by $Q$.

**Proposition 12.4** (Powers)**.** *Keep the foregoing notation. Suppose that the net $\{\alpha_x(Q)\}_{x\in P}$ converges to 1 in SOT. Then $\omega_\xi$ is absorbing for $\alpha$.*

*Proof.* Let $Q_x := \alpha_x(Q)$. We write $\omega$ in place of $\omega_\xi$. Consider a normal state $\rho$ of $B(\mathcal{H})$. Calculate as follows to observe that, for $A \in B(\mathcal{H})$,

$$\begin{aligned}
&|\rho\circ\alpha_x(A) - \omega(A)| \\
&= |\rho(\alpha_x(QAQ + QA(1-Q) + (1-Q)AQ + (1-Q)A(1-Q))) - \omega(A)| \\
&= |\omega(A)\rho(Q_x) + \rho(Q_x\alpha_x(A)Q_x^{\perp}) + \rho(Q_x^{\perp}\alpha_x(A)Q_x) + \rho(Q_x^{\perp}\alpha_x(A)Q_x^{\perp}) - \omega(A)| \\
&\leq |\omega(A)||\rho(Q_x^{\perp})| + 3||A||\rho(Q_x^{\perp})^{\frac{1}{2}} (\text{ by the Cauchy-Schwarz inequality for states}) \\
&\leq 4||A||\rho(Q_x^{\perp})^{\frac{1}{2}}.
\end{aligned}$$

Thus $||\rho\circ\alpha_x - \omega|| \leq 4\rho(1-Q_x)^{\frac{1}{2}}$. But $Q_x$ converges strongly to 1. Thus $\rho(Q_x) \to 1$. As a consequence, we obtain $||\rho\circ\alpha_x - \omega|| \to 0$. Hence $\omega$ is absorbing for $\alpha$. This completes the proof. □

Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$ and let $\xi \in \mathcal{H}$ be a unit vector. Suppose that $\omega_\xi$ is invariant. For $x \in P$, let $S_x : \mathcal{H} \to \mathcal{H}$ be the bounded operator defined by the equation

$$S_xA\xi = \alpha_x(A)\xi. \tag{12.6}$$

The fact that $\omega_\xi$ is invariant implies that $S_x$ is well defined. It is clear that $S_x \in E(x)$ where $E := \{E(x)\}_{x\in P}$ is the product system of $\alpha$. It is routine to verify that $S_xS_y = S_{xy}$ and $\langle S_x|S_x\rangle = 1$. In other words $\{S_x\}_{x\in P}$ is a unit for the product system $E$. Thus $E_0$-semigroups which are cocycle conjugate to one in standard form is always spatial.

**Proposition 12.5.** *Keep the foregoing notation. The following are equivalent.*

1. *The vector state $\omega_\xi$ is absorbing for $\alpha$.*
2. *The vector $\xi$ is cyclic, i.e. the increasing union $\bigcup_{x\in P} E(x)\xi$ is dense in $\mathcal{H}$.*

(3) *The net* $\{\alpha_x(Q)\}_{x \in P}$ *converges to* 1 *in SOT where* $Q$ *is the projection onto the one dimensional space spanned by* $\xi$.

*Proof.* The equivalence between (2) and (3) follows from Lemma 12.3. That (3) $\implies$ (1) follows from Prop. 12.4. Now suppose that (1) holds. Let $\{S_x\}_{x \in P}$ be the unit given by Eq. 12.6. First observe that $\{E(x)\xi : x \in P\}$ is an increasing family of closed subspaces. To see this, let $x, y \in P$ be such that $y = xa$ for some $a \in P$. Then $E(x)\xi = E(x)S_a\xi \subset E(y)\xi$.

Let $R$ be the projection onto the closure of $\bigcup_{x \in P} E(x)\xi$. Note that $E(x)$ leaves $Ran(R)$ invariant for every $x \in P$. Fix $x \in P$ and let $T \in E(x)$ be given. We claim that $T^*$ leaves $Ran(R)$ invariant. It suffices to show that $T^*$ leaves the dense subspace $\bigcup_{x \in P} E(x)\xi$ invariant. Let $\eta \in \bigcup_{x \in P} E(x)\xi$ be given. We can assume, without loss of generality, that there exists $y \geq x$ such that $\eta \in E(y)\xi$. Write $y = xa$. Then $T^*E(y) \subset E(a)$. Hence $T^*\eta \subset E(a)\xi \subset Ran(R)$. Thus, for every $x \in P$, $E(x)$ and $E(x)^*$ leaves $Ran(R)$ invariant. As a consequence, $E(x)$ and $E(x)^*$ commutes with $R$ for every $x \in P$.

Fix $x \in P$ and let $\{V_i\}_{i=1}^d$ be an orthonormal basis for $E(x)$. We have just seen that $V_i^*$ commutes with $R$ for every $i$. Thus $\alpha_x(R) = \sum_{i=1}^d V_iRV_i^* = (\sum_{i=1}^d V_iV_i^*)R = R$. As a consequence, $R$ is left invariant by $\alpha_x$ for every $x \in P$. But $\omega_\xi$ is absorbing for $\alpha$ which forces that $\alpha$ is pure. Hence $R = Id$. This completes the proof. □

Summarising, we have the following.

**Proposition 12.6.** *The* $E_0$*-semigroup* $\alpha$ *is in standard form if and only if there exists a unit vector* $\xi \in \mathcal{H}$ *such that*

(1) *for* $x \in P$, $\xi$ *is a common eigen vector for* $\{T^* : T \in E(x)\}$, *and*
(2) *the increasing union* $\bigcup_{x \in P} E(x)\xi$ *is dense in* $\mathcal{H}$.

We have seen that an $E_0$-semigroup in standard form is necessarily spatial, i.e. its product system has a unit. The natural question to ask is if $\alpha$ is a spatial $E_0$-semigroup, is $\alpha$ cocycle conjugate to one in standard form? The answer is yes.

Let $E$ be a product system and $S := \{S_x\}_{x \in P}$ be a unit such that $\langle S_x | S_x \rangle = 1$. Such units are called isometric units. Let $\phi$ be the representation of $E$ constructed using Arveson's inductive limit procedure. We use the notation developed at the end of Chapter 4. Thus $\mathcal{H}$ is the inductive limit of the Hilbert spaces $\{E(x)\}_{x \in P}$ where the connecting isometries $V_{y,x} : E(x) \to E(y)$ are given by $V_{y,x}(e) = eS_{x^{-1}y}$. Let $i_x : E(x) \to \mathcal{H}$ be the natural inclusion. Let $\alpha^S$ be the $E_0$-semigroup on $B(\mathcal{H})$ corresponding to the representation $\phi$. Pick $x \in P$ and let $\xi := i_x(S_x)$. Note that $\xi$ is independent of the choice of $x$ precisely because $S$ is a unit.

**Proposition 12.7.** *The* $E_0$*-semigroup* $\alpha^S := \{\alpha_x\}_{x \in P}$ *is in standard form.*

*Proof.* Let $x \in P$ and $e \in E(x)$ be given. Note that $\phi(S_x)\xi = \phi(S_x)i_y(S_y) = i_{xy}(S_xS_y) = i_{xy}(S_{xy}) = \xi$. Calculate as follows to observe that

$$\begin{aligned}\phi(e)^*\xi &= \phi(e)^*\phi(S_x)\xi \\ &= \langle S_x|e\rangle\xi.\end{aligned}$$

Thus $\xi$ is a common eigen vector for $\{\phi(e)^* : e \in E(x)\}$ for every $x \in P$.

Observe that for $e \in E(x)$, $\phi(e)\xi = \phi(e)i_e(\lambda) = \lambda i_x(e)$ where $\lambda$ is a scalar of modulus 1. Thus $\phi(E(x))\xi = i_x(E(x))$. By definition, the union $\bigcup_{x\in P} i_x(E(x))$ is dense in $\mathcal{H}$. The proof is completed by appealing to Prop. 12.6. □

Let $\alpha := \{\alpha_x\}_{x\in P}$ be an $E_0$-semigroup on $B(\mathcal{H})$. Suppose that $\alpha$ is in standard form with the invariant vector state $\omega_\xi$. Let $S := \{S_x\}_{x\in P}$ be the unit given by Eq. 12.6.

**Lemma 12.8.** *Keep the foregoing notation. The $E_0$-semigroups $\alpha$ and $\alpha^S$ are conjugate.*

*Proof.* Let $\mathcal{H}_\infty$ be the inductive limit of the Hilbert spaces $\{E(x)\}_{x\in P}$ where the connecting isometries $V_{y,x} : E(x) \to E(y)$ are given by $V_{y,x}(e) = eS_{x^{-1}y}$. Let $\xi_0 \in \mathcal{H}_\infty$ be defined by $\xi_0 = i_x(S_x)$. Define $U : \mathcal{H}_\infty \to \mathcal{H}$ by the equation $U(i_x(e)) = e\xi$. The fact that such an operator can be defined is justified by Exercise 10.2. It is clear that $U$ preserves the inner product. Since $\bigcup_{x\in P} E(x)\xi$ is total in $\mathcal{H}$, it follows that $U$ is a unitary.

Let $\phi : E \to B(\mathcal{H}_\infty)$ be the representation of $E$ associated to the $E_0$-semigroup $\alpha^S$. Fix $x \in P$ and $e \in E(x)$. For $f \in E(y)$, calculate as follows to observe that

$$\begin{aligned}U\phi(e)U^*f\xi &= U\phi(e)i_y(f) \\ &= Ui_{xy}(ef) \\ &= ef\xi \\ &= e(f\xi).\end{aligned}$$

Since $\{f\xi : f \in E(y), y \in P\}$ is total in $\mathcal{H}$, it follows that $U\phi(e)U^* = e$. This proves that $\alpha$ and $\alpha^S$ are conjugate. This completes the proof. □

The final question that requires answering is the following. Suppose $S$ and $T$ are two (isometric) units of a product system $E$. When is $\alpha^S$ conjugate to $\alpha^T$ ? To answer this, we need to introduce the notion of a gauge group of a product system.

**Definition 12.9.** *Let $E$ be a product system over $P$. The set of automorphisms of $E$ form a group under composition and is called the automorphism group or the gauge group of $E$. We denote the gauge group of $E$ by $Aut(E)$.*

For explicit computation of gauge groups of CCR flows, we refer the reader to [7], [3] and [4]. Note that the gauge group acts naturally on the set of units. Suppose

$u := \{u_x\}_{x \in P}$ is a unit and $\theta := \coprod_{x \in P} \theta_x$ is an element in $Aut(E)$, then $\{\theta_x u_x\}_{x \in P}$ is a unit.

Let $E$ be a product system over $P$ and let $S := \{S_x\}_{x \in P}$ and $T := \{T_x\}_{x \in P}$ be isometric units. Let $\alpha^S$ and $\alpha^T$ be the $E_0$-semigroups given by Arveson's inductive limit procedure. We decorate the Hilbert spaces, connecting isometries, the natural inclusions by $S$. For instance, the Hilbert space on which $\alpha^S$ acts will be denoted $\mathcal{H}^S$. Similarly for $T$. Let $\phi^S$ be the representation of $E$ on $\mathcal{H}^S$ corresponding to $S$ and $\phi^T$, the representation corresponding to $T$.

**Proposition 12.10.** *Keep the foregoing notation. The following are equivalent.*

1. *The $E_0$-semigroups $\alpha^S$ and $\alpha^T$ are conjugate.*
2. *There exists $\theta \in Aut(E)$ such that $\theta(S) = T$.*

*Proof.* Suppose that (1) holds. Let $U : \mathcal{H}^S \to \mathcal{H}^T$ be a unitary such that $\alpha_x^T = Ad(U) \circ \alpha_x^S \circ Ad(U^*)$. Let $\xi^S := i_x^S(S_x)$ and $\xi^T = i_x^T(T_x)$. Then $\omega_{\xi^S} \circ Ad(U^*)$ is absorbing for $\alpha^T$. But absorbing states are unique. Thus $\omega_{\xi^T} = \omega_{\xi^S} \circ Ad(U^*)$. This has the consequence that $U\xi^S$ is a scalar multiple of $\xi^T$. By rescaling, if necessary, we can assume that $U\xi^S = \xi^T$.

Define $\theta : E \to E$ as follows: for $x \in P$ and $e \in E(x)$, let $\theta_x(e) := (\phi^T)^{-1}(U\phi^S(e)U^*)$. Clearly $\theta$ is an automorphism of $E$. For $A \in B(\mathcal{H}^T)$ and $x \in P$, calculate as follows to observe that

$$\begin{aligned}
U\phi^S(S_x)U^*A\xi^T &= U\phi^S(S_x)U^*AU\xi^S \\
&= U\alpha_x^S(U^*AU)\xi^S \\
&= U\alpha_x^S(U^*AU)U^*\xi^T \\
&= \alpha_x^T(A)\xi^T \\
&= \phi^T(T_x)A\xi^T.
\end{aligned}$$

The above calculation implies that $\theta(S) = T$. Thus the implication (1) $\implies$ (2) is proved.

Now assume that (2) holds. Choose $\theta \in Aut(E)$ such that $\theta(S) = T$. Define $U : \mathcal{H}^S \to \mathcal{H}^T$ by the equation $Ui_x^S(e) = i_x^T(\theta_x(e))$. The map $U$ is well defined precisely because $\theta$ maps $S$ to $T$. It is clear that $U$ is a unitary. Let $x \in P$, $e \in E(x)$ be given. For $y \in P$

and $f \in E(y)$, calculate as follows to observe that

$$\begin{aligned} U\phi^S(e)U^* i_y^T(f) &= U\phi^S(e) i_y^S(\theta_y^{-1}(f)) \\ &= U i_{xy}^S(e\theta_y^{-1}(f)) \\ &= i_{xy}^T(\theta_{xy}(e\theta_y^{-1}(f)) \\ &= i_{xy}^T(\theta_x(e)f) \\ &= \phi^T(\theta(e)) i_y^T(f). \end{aligned}$$

Thus $U\phi^S(e)U^* = \phi^T(\theta(e))$ for every $e \in E(x)$. For $x \in P$, note that the intertwining space of $Ad(U) \circ \alpha_x^S \circ Ad(U)^*$ is $\{U\phi^S(e)U^* : e \in E(x)\}$ which equals $\{\phi^T(f) : f \in E\}$ which is the intertwining space of $\alpha_x^T$. Consequently, for $x \in P$,

$$\alpha_x^T = Ad(U) \circ \alpha_x^S \circ Ad(U^*).$$

Thus the implication (2) $\implies$ (1) is proved. This completes the proof. □

The following is an immediate corollary of Prop. 12.8 and Prop. 12.10.

**Corollary 12.11.** *Let $\alpha := \{\alpha_x\}_{x \in P}$ and $\beta := \{\beta_x\}_{x \in P}$ be $E_0$-semigroups. Suppose $\alpha$ and $\beta$ are in standard form. Suppose that the gauge group of $\alpha$ and $\beta$ acts transitively on the respective set of units. Then $\alpha$ is cocycle conjugate to $\beta$ if and only if $\alpha$ and $\beta$ are conjugate.*

**Remark 12.12.** *Tsirelson in* [37] *has constructed one parameter examples of $E_0$-semigroups whose gauge group does not act transitively on the set of its units.*

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